Classical Dynamics

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1. Vectors

1.1 Introduction

Definition 1 (Vectors). Vectors are mathematical objects with both **magnitude** and **direction**.

Geometrially, vectors can be thought of as arrows/direced line segments in space in space.

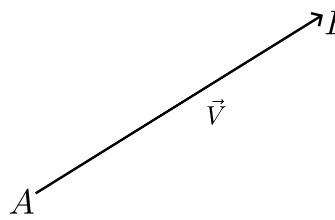


Figure 1.1: A Vector

Example (Examples of vectors). Here are some important examples of vectors

- The displacement of a particle is a vector.
- The velocity of a particle is a vector.
- The force acting on a particle is a vector.

Notation. Vectors can be denoted in 3 ways,

- Using **boldface notation**: $\boldsymbol{V}$
- Underlining: $\underline{V}$
- An arrow over the symbol: $\vec{V}$

1.2 Euclidean Three Space $\mathbb{E}^3$

Definition 2 (Euclidian Three Space). Euclidean Three Space is the set of all ordered triples of real numbers.

$$\mathbb{E}^3 = \{(x, y, z)|x, y, z \in \mathbb{R}\} \tag{1.1}$$

The **axes** of $\mathbb{E}^3$ are the x, y and z, i.e.

$$x = (x, 0, 0), y = (0, y, 0), z = (0, 0, z) \tag{1.2}$$

We orient the axis according to the **right hand rule**. This is shown in the following diagram:

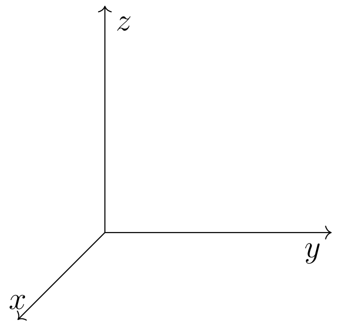


Figure 1.2: Axes in $\mathbb{E}^3$

> **Note.** We need to pick an **origin** and stay with it. We will use the origin $(0, 0, 0)$.

1.3 Vectors in $\mathbb{E}^3$

1.3.1 Distance in $\mathbb{E}^3$

Let P and $P^{'}$ be points in $\mathbb{E}^3$. And let $P = (x, y, z)$ and $P^{'} = (x^{'}, y^{'}, z^{'})$.

Definition 3 (Distance in $\mathbb{E}^3$). The **distance** between P and $P^{'}$ is defined as:

$$d(P, P^{'}) = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \tag{1.3}$$

This is illustrated in the following diagram

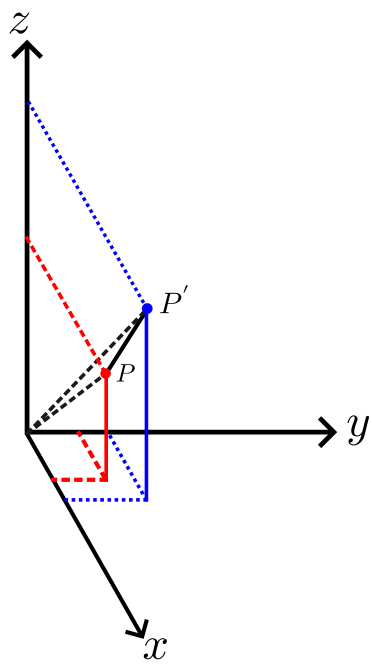


Figure 1.3: Distance in $\mathbb{E}^3$

1.3.2 Vectors in $\mathbb{E}^3$

Definition 4 (Vectors in $\mathbb{E}^3$). A **vector** in $\mathbb{E}^3$ is an ordered triple of real numbers.

$$\vec{v} = (v_1, v_2, v_3) \tag{1.4}$$

Notation. We can also represent vectors using **column notation**

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

1.4 Vector Algebra

1.4.1 Vector Magnitude

Definition 5 (Vector Magnitude). Let $\vec{v} = (v_1, v_2, v_3)$ be a vector in $\mathbb{E}^3$. The **magnitude** of $\vec{v}$ is defined as:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{1.5}$$

1.4.2 Vector Addition

Definition 6 (Vector Addition). Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{E}^3$. The **sum** of $\vec{v}$ and $\vec{w}$ is defined as:

$$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3) \tag{1.6}$$

Geometrically this can be seen as the **diagonal** of a paralleleogram. Geometrically it is clear that you get the same effect as travelling along $\vec{v}$ and then $\vec{u}$

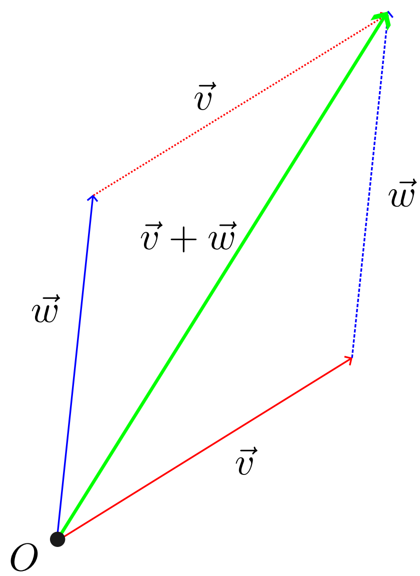


Figure 1.4: Vector Addition

Vector Addition Properties

Theorem 1 (Commutativity). Suppose $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{E}^3$.
If $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{E}^3$, then

$$\vec{v} + \vec{w} = \vec{w} + \vec{v} \tag{1.7}$$

Proof. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{E}^3$. Then

$$\begin{aligned}
\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} &= \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix} \\
&= \begin{bmatrix} w_1 + v_1 \\ w_2 + v_1 \\ w_3 + v_2 \end{bmatrix} \qquad \text{commutativity in } \mathbb{R} \\
&= \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{w} + \vec{v}
\end{aligned}$$

□

Theorem 2 (Associativity). Suppose $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{E}^3$.
If $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{E}^3$, then

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \tag{1.8}$$

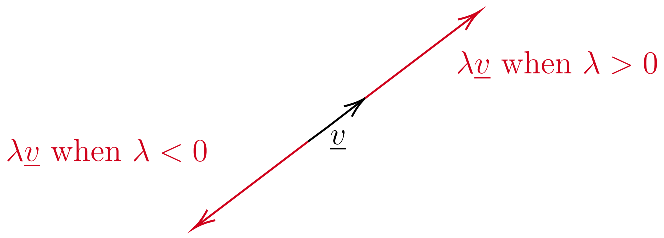
Proof. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{E}^3$. Then

$$\begin{aligned}
\vec{u} + (\vec{v} + \vec{w}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) &= \begin{bmatrix} u_1 + (v_1 + w_1) \\ u_2 + (v_2 + w_2) \\ u_3 + (v_3 + w_3) \end{bmatrix} \\
&= \begin{bmatrix} (u_1 + v_1) + w_1 \\ (u_2 + w_2) + v_2 \\ (u_2 + v_3) + w_2 \end{bmatrix} \qquad \text{commutativity in } \mathbb{R} \\
&= \begin{bmatrix} u_1 + v_1 \\ u_1 + v_1 \\ u_1 + v_1 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = (\vec{u} + \vec{v}) + \vec{w}
\end{aligned}$$

□

1.4.3 Scalar Multiplication

Vectors can be multiplied by scalars to get a new vector. This is called **scalar multiplication**. The direction of the new vector depends on the sign of the scalar.



Definition 7 (Scalar Multiplication). Let $\vec{v} = (v_1, v_2, v_3)$ be a vector in $\mathbb{E}^3$ and $\lambda \in \mathbb{R}$ be a scalar. The **scalar multiplication** of $\vec{v}$ and λ is defined as:

$$\lambda \vec{v} = (\lambda v_1, \lambda v_2, \lambda v_3) \tag{1.9}$$

Multiplying by a Scalar

Let $\vec{v}$ be a vector in $\mathbb{E}^3$ and λ be a scalar. Then:

- If $\lambda > 0$, then $\lambda \vec{v}$ is a vector in the same direction as $\vec{v}$ but with magnitude $\lambda \|\vec{v}\|$
- If $\lambda < 0$, then $\lambda \vec{v}$ is a vector in the opposite direction as $\vec{v}$ but with magnitude $|\lambda| \|\vec{v}\|$

Scalar Multiplication Properties

Theorem 3 (Distributivity over Scalar Multiplication). Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{E}^3$ and λ be a scalar. Then

$$\lambda(\vec{u} + \vec{v}) = \lambda \vec{u} + \lambda \vec{v} \tag{1.10}$$

Proof. Let $\vec{u}, \vec{v} \in \mathbb{E}^3$

$$\lambda(\vec{u} + \vec{v}) = \lambda \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} = \begin{bmatrix} \lambda(u_1 + v_1) \\ \lambda(u_2 + v_2) \\ \lambda(u_3 + v_3) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda u_1 + \lambda v_1 \\ \lambda u_2 + \lambda v_2 \\ \lambda u_3 + \lambda v_3 \end{bmatrix} = \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{bmatrix} + \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{bmatrix} = \lambda \vec{u} + \lambda \vec{v}$$

□

Theorem 4 (Associativity). Let $\vec{v}$ be a vector in $\mathbb{E}^3$ and λ, μ be scalars. Then

$$(\lambda\mu)\vec{v} = \lambda(\mu\vec{v}) \tag{1.11}$$

Proof. Let $\vec{v} \in \mathbb{E}^3$

$$(\lambda\mu)\vec{v} = (\lambda\mu)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (\lambda\mu)v_1 \\ (\lambda\mu)v_2 \\ (\lambda\mu)v_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda(\mu v_1) \\ \lambda(\mu v_2) \\ \lambda(\mu v_3) \end{bmatrix} = \lambda \begin{bmatrix} \mu v_1 \\ \mu v_2 \\ \mu v_3 \end{bmatrix} = \lambda(\mu\vec{v})$$

□

Theorem 5 (Distributivity over Vector Addition). Let $\vec{v}$ be a vector in $\mathbb{E}^3$ and λ, μ be scalars. Then

$$(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v} \tag{1.12}$$

Proof. Let $\vec{v}$ be a vector in $\mathbb{E}^3$

$$(\lambda + \mu)\vec{v} = (\lambda + \mu)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (\lambda + \mu)v_1 \\ (\lambda + \mu)v_2 \\ (\lambda + \mu)v_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda v_1 + \mu v_1 \\ \lambda v_2 + \mu v_2 \\ \lambda v_3 + \mu v_3 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{bmatrix} + \begin{bmatrix} \mu v_1 \\ \mu v_2 \\ \mu v_3 \end{bmatrix} = \lambda\vec{v} + \mu\vec{v}$$

□

Theorem 6 (Identity). Let $\vec{v}$ be a vector in $\mathbb{E}^3$. Then

$$1\vec{v} = \vec{v} \tag{1.13}$$

Proof. Let $\vec{v}$ be a vector in $\mathbb{E}^3$

$$1\vec{v} = 1(v_1, v_2, v_3) = (1v_1, 1v_2, 1v_3)$$

$$= (v_1, v_2, v_3) = \vec{v}$$

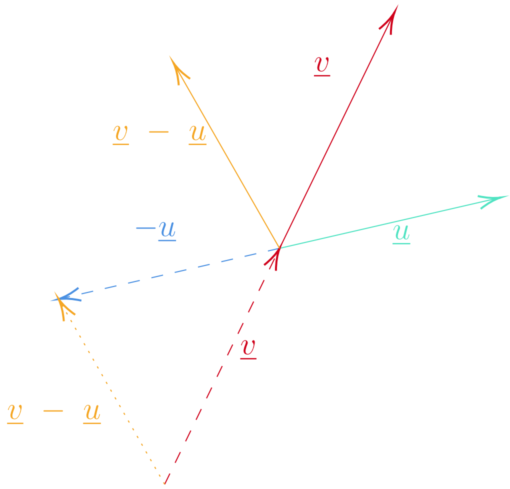
□

1.4.4 Vector Subtraction

Definition 8 (Vector Subtraction). Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{E}^3$. The **difference** of $\vec{v}$ and $\vec{w}$ is defined as:

$$\vec{v} - \vec{w} = \vec{v} + (-1)\vec{w} \tag{1.14}$$

Geometrically we can see this in the following diagram:



1.4.5 Unit Vectors

Definition 9 (Unit Vector). A **unit vector** is a vector with magnitude 1. The unit vector in the **direction** of $\vec{v}$ is denoted by $\hat{v}$. Unit vector is calculated by:

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} \tag{1.15}$$

1.5 Standard Basis

Standard basis vectors are also known as standard **unit vectors**. These are used to represent vectors in $\mathbb{E}^3$

Definition 10. The standard basis vectors are defined as follows:

$$\hat{i} = (1, 0, 0)$$
$$\hat{j} = (0, 1, 0)$$
$$\hat{k} = (0, 0, 1)$$

such that $\mid \hat{i} \mid = \mid \hat{j} \mid = \mid \hat{k} \mid$.

Any vector can be represented using standard basis vectors.

Suppose you are a given a vector $\vec{v} = (v_1, v_2, v_3)$. This can be represented as follows:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k} \tag{1.16}$$

Example. Let $\vec{v} = (2, 3, 4)$. Then,

$$\begin{aligned}\vec{v} &= 2\hat{i} + 3\hat{j} + 4\hat{k} \\ &= 2(1,0,0) + 3(0,1,0) + 4(0,0,1) \\ &= (2,0,0) + (0,3,0) + (0,0,4) \\ &= (2,3,4)\end{aligned}$$

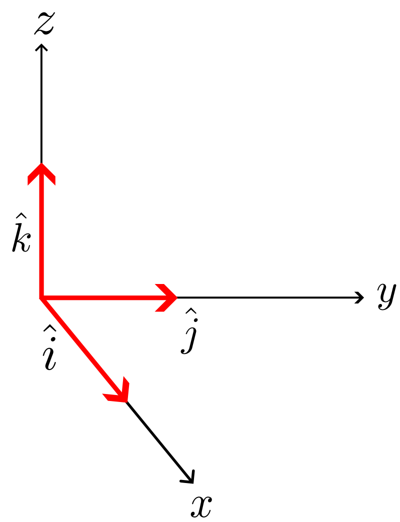


Figure 1.5: Standard Basis Vectors

Algebra with Standard Basis Vectors

Example. Let $\vec{v}$ and $\vec{w} \in \mathbb{E}^3$

$$\vec{v} \pm \vec{w} = \begin{bmatrix} v_1 \pm w_1 \\ v_2 \pm w_2 \\ v_3 \pm w_3 \end{bmatrix} = (v_1 \pm w_1)\hat{i} + (v_2 \pm w_2)\hat{j} + (v_3 \pm w_3)\hat{k}$$

Example. Let $\vec{v}$ and $\vec{w} \in \mathbb{E}^3$

$$\lambda\vec{v} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{bmatrix} = (\lambda v_1)\hat{i} + (\lambda v_2)\hat{j} + (\lambda v_3)\hat{k}$$

Note. The 0 vector is:

$$\underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Any vector $\vec{v} \in \mathbb{E}^3$ added to the 0 vector is itself:

$$\vec{v} + \underline{0} = \vec{v}$$

Here is an example of algebra with standard basis vectors:

Example. Let $\vec{v} = (2, 3, 4)$ and $\vec{w} = (1, 2, 3)$. Then,

$$\begin{aligned} \vec{v} + \vec{w} &= (2, 3, 4) + (1, 2, 3) \\ &= (2 + 1, 3 + 2, 4 + 3) \\ &= (3, 5, 7) \end{aligned}$$

Alternate Notation for Standard Basis Vectors

Notation. We can change notation for standard basis vectors as follows:

$$\hat{i} = \vec{e}_1 \qquad \hat{j} = \vec{e}_2 \qquad \hat{k} = \vec{e}_3$$

and therefore we can write:

$$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k} = \sum_{a=1}^{3} v_a \vec{e}_a$$

1.6 Position Vectors

Definition 11. A **position vector** is a vector that represents the position of a point in space relative to the origin, O.

Let any vector $\vec{v}$ be the position vector of a point P in space. Then, the coordinates of P are given by the components of $\vec{v}$:

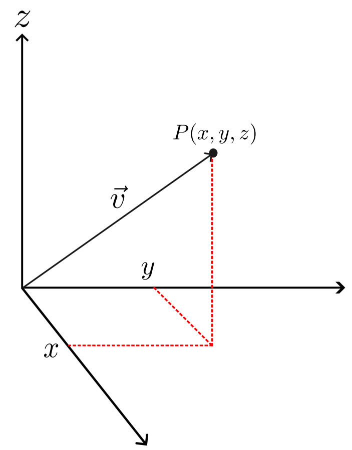


Figure 1.6: Position Vector

So the position vector of P is given by:

$$\vec{v} = \begin{bmatrix} x \\ y \\ j \end{bmatrix} = x\hat{i} + y\hat{j} + k\hat{k} \tag{1.17}$$

1.7 Scalar Product

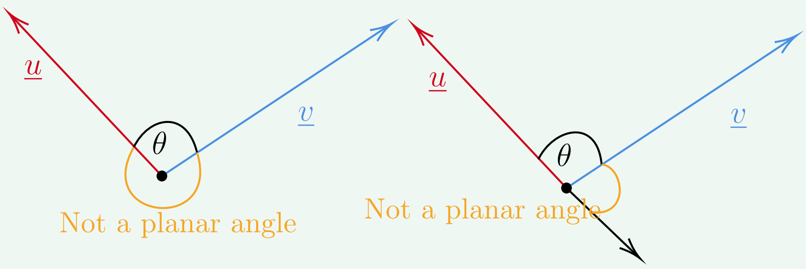
Scalar product is also known as dot product, is a function denoted by $\cdot$:

$$\cdot : \mathbb{E}^3 \times \mathbb{E}^3 \mapsto \mathbb{R}$$

i.e. it takes two vectors and returns a scalar.

Definition 12 (Planar Angle). Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{E}^3$ and $\theta \in \mathbb{R}$.

The **planar angle** between two vectors $\vec{v}$ and $\vec{w}$ is the angle θ between them in the plane spanned by $\vec{v}$ and $\vec{w}$.



Choose the planar angle θ such that

$$0 \leq \theta \leq \pi$$

.

Definition 13 (Scalar Product). Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{E}^3$ and $\theta \in \mathbb{R}$ be the planar angle between them.
Then, the scalar product of $\vec{v}$ and $\vec{w}$ is defined as:

$$\vec{v} \cdot \vec{w} = \mid \vec{v} \mid\mid \vec{w} \mid \cos\theta \tag{1.18}$$

Note. Two vectors do not lie in the same line, always in the same plane. By the convention, the angle $\theta \in [0, \pi] \Rightarrow 0 \leq \theta \leq \pi$.

1.7.1 Properties of Scalar Product

Theorem 7 (Commutative). Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{E}^3$. Then,

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \tag{1.19}$$

Proof. Since the planar angle θ is the same for both $\vec{v}$ and $\vec{w}$,

$$\begin{aligned}
\vec{v} \cdot \vec{w} &= \mid \vec{v} \mid\mid \vec{w} \mid \cos\theta \\
&= \mid \vec{w} \mid\mid \vec{v} \mid \cos\theta \\
&= \vec{w} \cdot \vec{v}
\end{aligned}$$

□

Theorem 8 (Orthogonal Vectors). Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{E}^3$. Then,

$$\vec{v} \cdot \vec{w} = 0 \Leftrightarrow \vec{v} \perp \vec{w} \tag{1.20}$$

Proof. When $\vec{v} \perp \vec{w}$, the planar angle $\theta = \frac{\pi}{2}$. Therefore,

$$\begin{aligned}
\vec{v} \cdot \vec{w} &= |\vec{v}||\vec{w}| \cos\theta \\
&= |\vec{v}||\vec{w}| \cos\frac{\pi}{2} \\
&= |\vec{v}||\vec{w}| \cdot 0 \\
&= 0
\end{aligned}$$

i.e. $\vec{v}$ and $\vec{w}$ are orthogonal. □

Theorem 9 (Distributivity over scalar multiplication). Let $\vec{v}, \vec{w} \in \mathbb{E}^3$ and $\lambda \in \mathbb{R}$. Then,

$$\lambda(\vec{v} \cdot \vec{w}) = (\lambda\vec{v}) \cdot \vec{w} = \vec{v} \cdot (\lambda\vec{w}) \tag{1.21}$$

Theorem 10 (Distributivity over Addition). Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{E}^3$. Then,

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \tag{1.22}$$

Note. Properties of scalar product for standard basis vectors:

$$\begin{aligned}
\hat{i} \cdot \hat{i} &= 1 \\
\hat{j} \cdot \hat{j} &= 1 \\
\hat{k} \cdot \hat{k} &= 1 \\
\hat{i} \cdot \hat{j} &= 0 \\
\hat{i} \cdot \hat{k} &= 0 \\
\hat{j} \cdot \hat{k} &= 0
\end{aligned}$$

1.7.2 Scalar Product in terms of Components

Theorem 11. Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be two vectors in $\mathbb{E}^3$. Then,

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{3} v_i w_i = v_1 w_1 + v_2 w_2 + v_3 w_3 \tag{1.23}$$

Proof. Let $\vec{v}, \vec{w} \in \mathbb{E}^3$. Then,

$$\vec{v} \cdot \vec{w} = (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \cdot (w_1\hat{i} + w_2\hat{j} + w_3\hat{k})$$

$$= v_1 \cdot (w_1\hat{i} + w_2\hat{j} + w_3\hat{k}) + v_2 \cdot (w_1\hat{i} + w_2\hat{j} + w_3\hat{k}) + v_3 \cdot (w_1\hat{i} + w_2\hat{j} + w_3\hat{k})$$

$$= v_1w_1\hat{i} \cdot \hat{i} + \cancel{v_1w_2\hat{i} \cdot \hat{j}} + \cancel{v_1w_3\hat{i} \cdot \hat{k}} + \cancel{v_2w_1\hat{j} \cdot \hat{i}} + v_2w_2\hat{j} \cdot \hat{j} + \cancel{v_2w_3\hat{j} \cdot \hat{k}}$$
$$+ \cancel{v_3w_1\hat{k} \cdot \hat{i}} + \cancel{v_3w_2\hat{k} \cdot \hat{j}} + v_3w_3\hat{k} \cdot \hat{k}$$

$$= v_1w_1 + v_2w_2 + v_3w_3$$

$$= \sum_{i=1}^{3} v_iw_i = v_1w_1 + v_2w_2 + v_3w_3$$

□

1.7.3 Using Scalar Product to find the length of a vector

We can also use scalar product to find the length of a vector.

Theorem 12. Let $\vec{v} \in \mathbb{E}^3$. Then,

$$\mid \vec{v} \mid = \sqrt{\vec{v} \cdot \vec{v}} \tag{1.24}$$

Proof. Let $\vec{v} \in \mathbb{E}^3$. Then,

$$\mid \vec{v} \mid = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$= \sqrt{v_1v_1 + v_2v_2 + v_3v_3}$$

$$= \sqrt{\vec{v} \cdot \vec{v}}$$

□

1.7.4 Using Scalar Product to find the angle between two vectors

Theorem 13. Let $\vec{v}, \vec{w} \in \mathbb{E}^3$. Then, the planar angle θ between $\vec{v}$ and $\vec{w}$ is given by:

$$\theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\mid \vec{v} \mid\mid \vec{w} \mid}\right) \tag{1.25}$$

Proof. Let $\vec{v}, \vec{w} \in \mathbb{E}^3$. Then,

$$\vec{v} \cdot \vec{w} = \mid \vec{v} \mid\mid \vec{w} \mid \cos\theta$$

$$= v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$\Rightarrow \cos\theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\mid \vec{v} \mid\mid \vec{w} \mid}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\mid \vec{v} \mid\mid \vec{w} \mid}\right)$$

□

Note. Some basic properties of scalar product:

1. If $\vec{v} \cdot \vec{w} = 0$, then $\theta = \frac{\pi}{2}$.
2. If $\vec{v} \cdot \vec{w} > 0$, then $\theta \in [0, \frac{\pi}{2})$.
3. If $\vec{v} \cdot \vec{w} < 0$, then $\theta \in (\frac{\pi}{2}, \pi]$.

1.8 Cross Product

Cross Product also known as **Vector Product** is a function denoted by

$$\times : \mathbb{E}^3 \times \mathbb{E}^3 \mapsto \mathbb{E}^3$$

i.e. it a binary operator on 2 vectors $\hat{t}$ returns a vector

Motivation for Vector

Given 2 non-zero vectors $\vec{u}$ and $\vec{v}$, construct a new vector say $\vec{w}$ such that it is **orthoogonal** to *both* $\vec{u}$ and $\vec{v}$

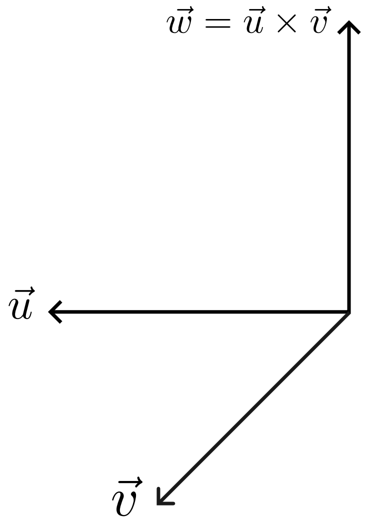


Figure 1.7: Cross Product

Definition 14. Given 2 vectors $\vec{u}$ and $\vec{v} \in \mathbb{E}^3$, the **cross product** of $\vec{u}$ and $\vec{v}$ is the vector $\vec{w}$ of **length**

$$\|\vec{w}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta \tag{1.26}$$

where θ is the **planar angle between** $\vec{u}$ and $\vec{v}$ and **direction** given by the **right hand rule**

We can determine the direction of $\vec{w}$ by using the **right hand rule** as shown in Figure 1.8

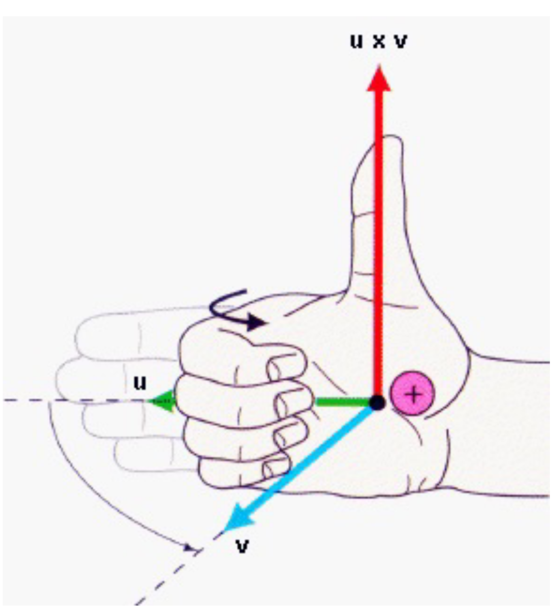


Figure 1.8: Cross Product Direction

1.8.1 Properties of Scalar Product

These properties can be seen as a consequence of the right and rule.

Theorem 14 (Anti-Commutativity). Let $\vec{v}, \vec{w} \in \mathbb{E}^3$ Then

$$\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v}) \tag{1.27}$$

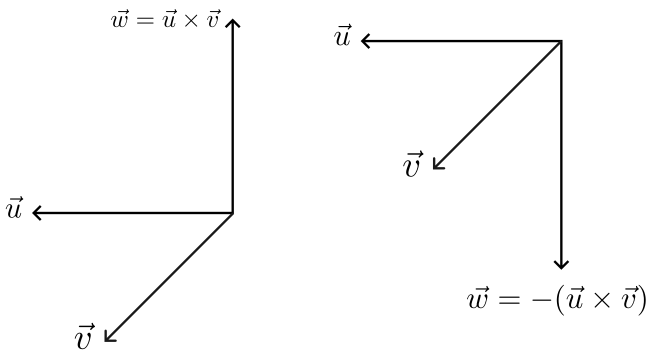


Figure 1.9: Cross Product Anti-Commutativity

Theorem 15 (Distributivity). Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{E}^3$ Then

$$\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w}) \tag{1.28}$$

Theorem 16 (Multiplication by Scalar). Let $\vec{u}, \vec{v} \in \mathbb{E}^3$ and $\lambda \in \mathbb{R}$ Then,

$$\lambda(\vec{u} \times \vec{v}) = (\lambda\vec{u}) \times \vec{v} = \vec{u} \times (\lambda\vec{v}) \tag{1.29}$$

Note. Properties of cross product on standard basis vectors

$$\begin{aligned}
\hat{i} \times \hat{j} &= \hat{k} \\
\hat{j} \times \hat{k} &= \hat{i} \\
\hat{k} \times \hat{i} &= \hat{j} \\
\hat{j} \times \hat{i} &= -\hat{k} \\
\hat{k} \times \hat{j} &= -\hat{i} \\
\hat{i} \times \hat{k} &= -\hat{j}
\end{aligned}$$

Note. The cross product is 0 when 2 vectors are **parallel**.
If $\vec{v} = \lambda\vec{u}$, then

$$\begin{cases} 0 & \text{if } \lambda > 0 \\ 1 & \text{if } \lambda < 0 \end{cases}$$

Since $\sin 0 = \sin \pi = 0$, we get

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta = 0$$

And hence

$$\vec{u} \times \vec{v} = 0$$

And therefore we can derive the following properties about standard basis vectors

$$\begin{aligned}
\hat{i} \times \hat{i} &= 0 \\
\hat{j} \times \hat{j} &= 0 \\
\hat{k} \times \hat{k} &= 0
\end{aligned}$$

1.8.2 Co-Ordinate Version of Cross Product

Theorem 17 (Co-Ordinate formula for Cross Product). Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{E}^3$. Then

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k} \tag{1.30}$$

Proof. Let $\vec{u}, \vec{v} \in \mathbb{E}^3$. Then

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) \times (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \\
&= (u_1\hat{i} \times v_1\hat{i}) + (u_1\hat{i} \times v_2\hat{j}) + (u_1\hat{i} \times v_3\hat{k}) + (u_2\hat{j} \times v_1\hat{i}) + (u_2\hat{j} \times v_2\hat{j}) + (u_2\hat{j} \times v_3\hat{k}) \\
&\quad + (u_3\hat{k} \times v_1\hat{i}) + (u_3\hat{k} \times v_2\hat{j}) + (u_3\hat{k} \times v_3\hat{k}) \\
&= \cancel{u_1v_1\hat{i} \times \hat{i}} + (u_1v_2\hat{i} \times \hat{j}) + (u_1v_3\hat{i} \times \hat{k}) + (u_2v_1\hat{j} \times \hat{i}) + \cancel{u_2v_2\hat{j} \times \hat{j}} + (u_2v_3\hat{j} \times \hat{k}) \\
&\quad + (u_3v_1\hat{k} \times \hat{i}) + (u_3v_2\hat{k} \times \hat{j}) + \cancel{u_3v_3\hat{k} \times \hat{k}} \\
&= (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}
\end{aligned}$$

□

Note. We can also write the cross product as

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{1.31}$$

Note. Showing that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

$$\begin{aligned}
\vec{u} \cdot (\vec{u} \times \vec{v}) &= u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) \\
&= u_1u_2v_3 - u_1u_3v_2 + u_2u_3v_1 - u_2u_1v_3 + u_3u_1v_2 - u_3u_2v_1 \\
\Rightarrow \vec{u} \cdot (\vec{u} \times \vec{v}) &= 0
\end{aligned}$$

and hence orthogonal. Proof similat for the other one.

1.9 Kronecker-Delta

As shown before, the properties of the **scalar product**, the **orthonormal basis** vectors have the following properties

$$\underline{e}_1 \cdot \underline{e}_2 = 0 = \underline{e}_1 \cdot \underline{e}_3 = \underline{e}_2 \cdot \underline{e}_3$$

and

$$\underline{e}_1 \cdot \underline{e}_1 = 1 = \underline{e}_2 \cdot \underline{e}_2 = \underline{e}_3 \cdot \underline{e}_3$$

We can abbreviate the definition using the **Kronecker-Delta**

Definition 15 (Kronecker-Delta). Let $a, b \in \{1, 2, 3\}$. Then we can write:

$$\underline{e_a} \cdot \underline{e_b} = \begin{cases} 1 & \text{if } a = b = 1, 2, 3 \\ 0 & \text{if } a \neq b \end{cases} = \delta_{ab} \tag{1.32}$$

This will also be useful for calculating scalar product

1.9.1 Scalar product using Kronecker-Delta

Theorem 18 (Scalar Product using Kroncker-Delta). Let $\vec{a}, \vec{b} \in \mathbb{E}^3$. Then

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_k b_k \tag{1.33}$$

Proof.

$$\underline{a} \cdot \underline{b} = \left(\sum_{k=1}^{3} a_k \underline{e_k}\right) \cdot \left(\sum_{l=1}^{3} b_l \underline{e_l}\right)$$

$$= \sum_{k,\ l} a_k\ b_l\ \underline{e_k} \cdot \underline{e_l}$$

$$= \sum_{k,\ l} a_k\ b_l\ \delta_{kl}$$

Now by the definition of Kronecker-Delta 1.32, it is 0 for all cases except when $k = l$. where it has a value of **1** So the summation becomes:

$$\underline{a} \cdot \underline{b} = \sum_{k=1}^{3} a_k\ b_k$$

□

1.10 Levi-Civita

We can represent the cross product using **Levi-Civita Symbol**

Definition 16 (Levi-Civita). Let $a, b, c \in 1, 2, 3$. Then we write:

$$\varepsilon_{abc} = \begin{cases} 0 & \text{if } a = b = c \text{ or more generally} a, b, c \text{is } \textbf{not} \text{ permutation of } 1, 2, 3 \\ +1 & \text{if } a, b, c \text{ is an } \textbf{even} \text{ permuation of } 1, 2, 3 \\ -1 & \text{if } a, b, c \text{ is an } \textbf{odd} \text{ permuation of } 1, 2, 3 \end{cases} \tag{1.34}$$

Note. Value of ε_{abc} depends on the **parity** of the permutation.

1.10.1 Cross product of Orthonoarmal Basis Using Levi-Civita

Definition 17. Then we can write the cross product **Orthonoarmal Basis Vectors** $\underline{e_1}, \underline{e_2}, \underline{e_3}$ in the following way

$$\underline{e_a} \times \underline{e_b} = \sum_{c=1}^{3} \varepsilon_{abc} \underline{e_c} \tag{1.35}$$

1.10.2 Cross Product of Vectors in Levi-Civita Notation

Theorem 19 (Cross Product using Levi-Civita Notation)**.** Let $\vec{a}, \vec{b} \in \mathbb{E}^3$. Then

$$\vec{a} \times \vec{b} = \sum_{m=1}^{3} (\vec{a} \times \vec{b})_m \underline{e_m} \tag{1.36}$$

where $(\vec{a} \times \vec{b})_m$ is the **mth component**

$$(\vec{a} \times \vec{b})_m = \sum_{k,\ l} \varepsilon_{klm} a_k b_l \tag{1.37}$$

Proof. Let

$$\underline{a} = a_k \underline{e_k} = \sum_{k=1}^{3} a_k\ \underline{e_k} \qquad \underline{b} = b_l \underline{e_l} = \sum_{l=1}^{3} b_l\ \underline{e_l}$$

Observe the use of **Eintein's Notation** (see below) and observe that

$$\underline{a} \times \underline{b} = \left(\sum_{k=1}^{3} a_k \underline{e_k}\right) \times \left(\sum_{l=1}^{3} b_l \underline{e_l}\right)$$

$$= \sum_{k,\ l} a_k\ b_k\ \underline{e_k} \times \underline{e_l}$$

And therefore by 1.35, we can rewrite it in the following way

$$= \sum_{k,\ l,\ m} a_k\ b_l\ \varepsilon_{klm}\ \underline{e_m}$$

Define the **mth component** as

$$(\vec{a} \times \vec{b})_m = \sum_{k,\ l} \varepsilon_{klm} a_k b_l$$

and hence we get 1.36 □

1.11 Einstein's Notation

Definition 18 (Einstein's Notation). If **two** indicies are **repeated**, then they are **summed** and we can **suppress** the summation

1.11.1 Scalar Product using Einstein Convention

Example.

$$\sum_{i=1}^{3} a_k b_k = a_k b_k = \delta_{kl} a_k b_l$$

Here the **repeated index** is k

1.11.2 Cross Product using Einstein Convention $\left(\underline{e}_a \times \underline{e}_b = \varepsilon_{abc} \underline{e}_c\right)$

Example. Here the **repeated index** is k and l

repeated index is k and l

$$(\underline{a} \times \underline{b})_m = \sum_{k,l} \varepsilon_{klm} a_k b_l = \varepsilon_{klm} a_k b_l$$

1.12 Triple Scalar Product

Theorem 20 (Triple Scalar Product). Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{E}^3$. Then

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \varepsilon_{pqr} a_p b_q c_r \tag{1.38}$$

Note. We have used **Einstein's Notation** for Scalar and Vector Product as well as Levi-Civita Notation 1.34

Proof.

$$\begin{aligned} \underline{a} \cdot (\underline{b} \times \underline{c}) &= a_p \, (\underline{b} \times \underline{c})_p \\ &= a_p \, \varepsilon_{pqr} \, b_q \, c_r \\ &= \varepsilon_{pqr} \, a_p \, b_q \, c_r \end{aligned}$$

□

Note. Although not mathematically valid, we can use the **determinant method**

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

1.13 Useful Properties of Kronecker-Delta and Levi-Civita

Theorem 21.

$$\varepsilon_{pqr}\varepsilon_{ruv} = (\delta_{pu}\delta_{qv} - \delta_{pv}\delta_{qu}) \tag{1.39}$$

Note. These are useful identity/trick to remember (remeber the use of **Einstein's Notation**)

$$\delta_{aa} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\delta_{ab}\delta_{bc} = \delta_{a1}\delta_{1c} + \delta_{a2}\delta_{2c} + \delta_{a3}\delta_{3c} = \begin{cases} 1 & \text{if } a = c \\ 0 & \text{if } a \neq c \end{cases} = \delta_{ac}$$

$\delta_{ab} n_b = n_a$

1.14 Triple Vector Product

Theorem 22 (Triple Vector Product)**.** Let $\underline{a}, \underline{b}, \underline{c} \in \mathbb{E}^3$

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b}) \tag{1.40}$$

Proof. Taking three vectors $\underline{a}, \underline{b}$ and $\underline{c}$, we *calculate* the **pth component** first:

$$[\underline{a} \times (\underline{b} \times \underline{c})]_p = \varepsilon_{pqr}\ a_q(\underline{b} \times \underline{c})_r$$

$$= \varepsilon_{pqr}\ a_q\ \varepsilon_{ruv}\ b_u\ c_v$$

using identity 1.39, we get

$$[\underline{a} \times (\underline{b} \times \underline{c})]_p = (\delta_{pu}\delta_{qv} - \delta_{pv}\delta_{qu})a_q b_u c_v$$

Use the explanation below for completion □

$$[\underline{a}\times(\underline{b}\times\underline{c})]_p = (\delta_{pu}\delta_{qv} - \delta_{pv}\delta_{qu})\, a_q b_u c_v$$

$$= \underbrace{\delta_{pu} b_u}_{\substack{\text{repeated}\\ \text{index :}\\ \text{Einstein}\\ \text{Convention}}}\ \underbrace{\delta_{qv} a_q c_v}_{\substack{\text{repeated}\\ \text{index :}\\ \text{Einstein}\\ \text{Convention}}} - \underbrace{\delta_{pv} c_v}_{\substack{\text{repeated}\\ \text{index ;}\\ \text{Einstein}\\ \text{Notation}}}\ \underbrace{\delta_{qu} a_q b_u}_{\substack{\text{repeated}\\ \text{index}}}$$

$$= \underbrace{\sum_{u=1}^{3}\delta_{pu} b_u}_{\substack{\shortparallel \\ \delta_{up} = \begin{cases} 1 & p=u \\ 0 & p\neq u\end{cases}}}\ \underbrace{\sum_{q,v}^{3}\delta_{qv} a_q c_v}_{\substack{\shortparallel\\ \text{scalar}\\ \text{product}\\ \text{defn}}} - \underbrace{\sum_{v=1}^{3}\delta_{pv} c_v}_{\substack{\shortparallel \\ \delta_{pv} = \begin{cases} 1 & p=v \\ 0 & p\neq v\end{cases}}}\ \underbrace{\sum_{qu}^{3}\delta_{qu} a_q b_u}_{\substack{\shortparallel\\ \text{scalar product}\\ \text{defn}}}$$

$$= b_p a_q c_q - c_p a_q b_q \quad \text{(Scalar product defn)}$$

$$= b_p(\underline{a}\cdot\underline{c}) - c_p(\underline{a}\cdot\underline{b})$$

This is true for all components $p \in \{1,2,3\}$

$$\Rightarrow \boxed{\underline{a}\times(\underline{b}\times\underline{c}) = \underline{b}(\underline{a}\cdot\underline{c}) - \underline{c}(\underline{a}\cdot\underline{b})}$$

NOT a dot product

Figure 1.10: Triple Vector Product Proof

1.15 Vector Equation of Lines

Definition 19 (Vector Equation of Lines). The **position vector** of $\underline{x}$ of an arbitrary point P(x,y,z) on the line in terms of $\underline{p}$ and $\underline{v}$

$$\underline{x} = \underline{p} + t\underline{v} \quad \text{for } t \in \mathbb{R} \tag{1.41}$$

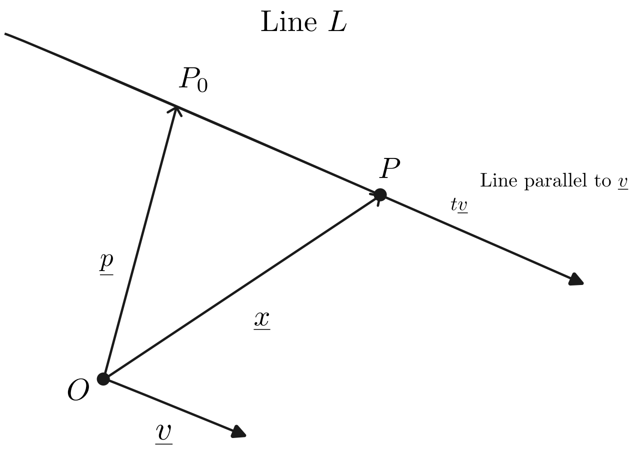


Figure 1.11: Vector Line

1.15.1 Parametric Equation of a Line

Note. Every point $\underline{x}$ can be written as

$$\underline{x} = x\hat{i} + y\hat{j} + z\hat{k}$$

Therefore we can form a parametric equation of a line

Definition 20 (Parametric Equation of a Line).

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases} \quad \text{for } t \in \mathbb{R} \tag{1.42}$$

1.15.2 Vector Equation of Line going through 2 points

Definition 21. Let P and Q be two points on the line and let their position vectors be $\underline{p}$ and $\underline{q}$ repectively. Then the **direction vector** is:

$$\vec{PQ} = \underline{q} - \underline{p}$$

and the vector equation line is

$$\underline{x} = +t(\underline{q} - \underline{p}) \quad \text{for } t \in \mathbb{R} \tag{1.43}$$

The following diagram depicts this:

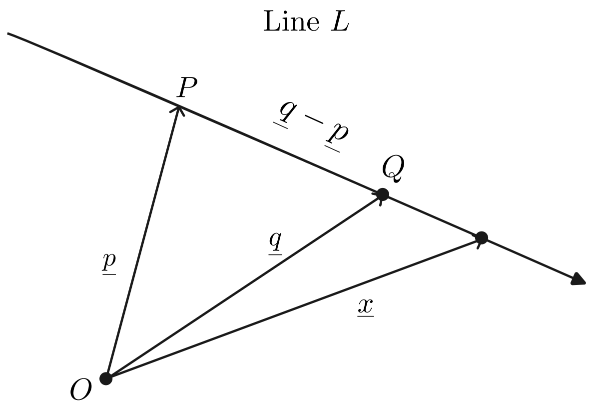


Figure 1.12: Vector Line

1.16 Vector Equation of Planes

Definition 22 (Vector Equation of Planes). Given a point P with **position vector** $\underset{\sim}{p}$ and 2 vectors **not** lying on the same line i.e. **not collinear**, then there is a plane that passes through P parallel to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$

The position vector of an *arbitrary* point x is

$$\underset{\sim}{x} = \underset{\sim}{p} + s\underset{\sim}{u} + t\underset{\sim}{v} \quad \text{for } s, t \in \mathbb{R} \tag{1.44}$$

This is known as the **plane spanned** by $\underset{\sim}{u}$ and $\underset{\sim}{v}$ going through P

1.16.1 Parametric Equation of a line

Definition 23.

$$\begin{cases} x = x_0 + su_1 + tv_1 \\ y = y_0 + su_2 + tv_2 \\ z = z_0 + su_3 + tv_3 \end{cases} \quad \text{for } s, t \in \mathbb{R} \tag{1.45}$$

1.16.2 Vector Equation of Planes using 3 Points

Definition 24. Given 3 non-linear points P, Q and R with **position vectors** $\underset{\sim}{p}, \underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively. Note that the vectors

$$(\underset{\sim}{p} - \underset{\sim}{r}) \quad \text{and} \quad (\underset{\sim}{q} - \underset{\sim}{r})$$

are two **direction vectors parallel to the plane**. Then taking r as the *starting point*, the equation of the plane becomes

$$\underset{\sim}{x} = \underset{\sim}{r} + s(\underset{\sim}{p} - \underset{\sim}{r}) + t(\underset{\sim}{q} - \underset{\sim}{r}) \quad \text{for } s, t \in \mathbb{R} \tag{1.46}$$

1.16.3 Normal Vector to a Plane

Another way to define a plane is by noticing that (in 3d) there is exactly one line which is perpendicular to the plane. A vector parallel to this line is called a **normal vector**.

Note (Unit Normal). If the **normal vector** has unit length, then it called a ***unit normal***
Thus if $\hat{n}$ is a unit normal to a plane then there is **exactly one other** unit normal to the plane namely $-\hat{n}$

Definition 25 (Equation of Plane using Normal Vector). If $\underline{p}$ is a **position vector** of a *known* point in the plane and $\underline{x}$ is any **arbitary point on the plane**, then

$$(\underline{x} - \underline{p})$$

is parallel to the plane and thus **orthogonal** to the normal. Therefore equation of a plane can be given as

$$(\underline{x} - \underline{p}) \cdot \underline{n} = 0 \qquad \text{(scalar product)} \tag{1.47}$$

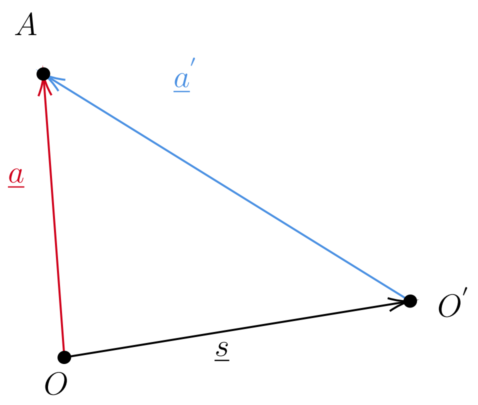
or multipying out the **Scalar Product**

$$\underline{x} \cdot \underline{n} = \underline{p} \cdot \underline{n}$$

1.17 Change of Axes

1.17.1 Change of Origin

Consider the following diagrams:



Here we have 2 vectors

- $\overrightarrow{OO}$
- $\overrightarrow{O^{'}O}$

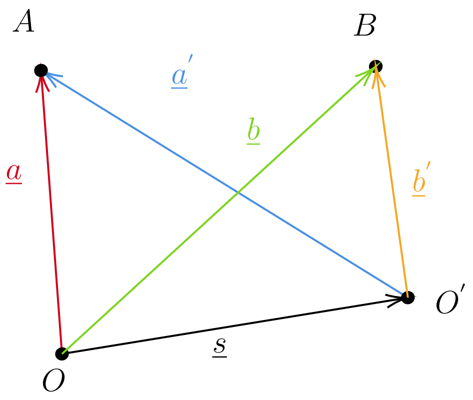
Shift of Origin

The **shift** of origin is represented by $\underline{s}$ is *relative* to O, then

$$\underline{u} = \underline{a} = \underline{s} + \underline{a}^{'} \Rightarrow \underline{a}^{'} = \underline{a} - \underline{s}$$

Note. $\underline{s}$ **could depend on time**

Note. Consider the following diagram

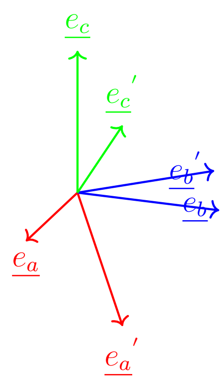


Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ represent vectors to A and B relative to O. Similarly let $\underset{\sim}{a}'$ and $\underset{\sim}{b}'$ represent vectors relative to O'. Then:

$$\underset{\sim}{a}' - \underset{\sim}{b}' = (\underset{\sim}{a} - \underset{\sim}{s}) - (\underset{\sim}{b} - \underset{\sim}{s}) = \underset{\sim}{a} - \underset{\sim}{b}$$

As we can see the **displacement** from $\underset{\sim}{a}$ to $\underset{\sim}{b}$ is the **same** as the **displacement** from $\underset{\sim}{a}'$ to $\underset{\sim}{b}'$.

1.17.2 Shifting and Changing Unit Vectors



Consider 2 sets of orthogonal unit vectors $\underset{\sim}{e_a}$ and $\underset{\sim}{e_a}'$ where $a \in \{1, 2, 3\}$.

We *can say* that each of the $\underset{\sim}{e_a}'$ is a **linear combination** of each if the $\underset{\sim}{e_a}$. So we can say that for a **matrix** R_{ab}

$$\underset{\sim}{e_a}' = R_{ab}\, \underset{\sim}{e_b} \tag{$*$}$$

Since the index b is **repeated twice**, we use the **Einstein Convention**. This is equivalent to:

$$\underset{\sim}{e_a}' = R_{a1}\, \underset{\sim}{e_1} + R_{a2}\, \underset{\sim}{e_2} + R_{a3}\, \underset{\sim}{e_3}$$

We **need to work out** that R_{ab} is.

First we *require* from the definition of Kronecker Delta 1.32:

$$\begin{aligned}\delta_{ac} &= \underline{e_a^{'}} \cdot \underline{e_c^{'}} \\ &= R_{ab}\ \underline{e_b} \cdot R_{cd}\ \underline{e_d} && \text{from equation } (*) \\ &= R_{ab}R_{cd}\ \underline{e_b} \cdot \underline{e_d} \\ &= R_{ab}R_{cd}\ \delta_{bd}\end{aligned}$$

Therefore we get:

$$\delta_{ac} = R_{ab}R_{cd}\ \delta_{bd}$$

By again using the **Einstein's Notation** the index b and d is **repeated twice**. Now in the RHS , δ_{bd} is **1** only when $\boldsymbol{d = b}$. Hence:

$$\delta_{ac} = R_{ab}R_{cd}\ \delta_{bd}$$

$$\Rightarrow \delta_{ac} = R_{ab}R_{cb}$$

In the expression $\delta_{ac} = R_{ab}R_{cb}$ it is **not** quite matrix multiplication since the **columns of the first** b is **not equal** to the *row of the second* c . Therefore we **transpose the matrix** and we get the following:

$$\delta_{ac} = R_{ab}R_{cb} = R_{ab}(R^T)_{bc} = (RR^T)_{ac}$$

$$\Rightarrow \delta_{ac} = (RR^T)_{ac}$$

Since δ_{ac} is the Kronecker Delta 1.32/identity matrix, we can say that:

$$RR^T = \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence R is an **orthogonal** matrix. Also note that

$$\begin{aligned}\det(\mathbb{1}) = 1 &= \det(RR^T) \\ &= \det(R)\det(R^T) && \text{property of det function} \\ &= (\det(R))^2 && \text{since } \det(R) = \det(R^T)\end{aligned}$$

Hence we get the expression:

$$1 = (\det(R))^2$$

$$\Rightarrow \det(R) = 1$$

Because R is **continuously connected** to the identity matrix we chose **+1**. Therefore **any matrix** R with the property:

$$\det(R) = 1$$

is a valid matrix.

The components of a vector are also related by the orthogonal matrix

$$x'_a \underline{e}'_a = x'_a R_{ab} \underline{e}_b = x_b \underline{e}_b$$

$$\Rightarrow \boxed{x_b = x'_a R_{ab}}$$

Note that

$$x'_a x'_a = R_{ab} x_b R_{ac} x_c = (R^T R)_{cb} x_b x_c = x_b x_b$$

$$\Rightarrow \boxed{x'_a x'_a = x_b x_b}$$ (Scalar product)

(i) If the rotation of axes is constant, then unit vector in each time frame are constant and a position vector is given by

$$\boxed{\underline{r} = x'_a \underline{e}'_a = x_a \underline{e}_a}$$

Velocity can be calculated differentiating w.r.t time

$$\frac{d\underline{r}}{dt} = \frac{dx'_a}{dt} e'_a = \frac{dx_a}{dt} \underline{e}_a$$

(ii) If rotation of axes is <u>not</u> independant of time (not constant)

$$\underline{e}'_a = R_{ab}(t) \underline{e}_b$$

Differentiating

$$\dot{\underline{e}}'_a = \dot{R}_{ab} \underline{e}_b = \dot{R}_{ab} R^T_{bc} \underline{e}_c$$

<u>Note</u>:

Since R is orthogonal, $RR^T = \mathbb{1}$. In index notation

$$R_{ab} R^T_{bc} \equiv R_{ab} R_{cb} = \delta_{ac}$$

Differentiating this (w.r.t time) (product rule)

$$\dot{R}_{ab}R_{cb} + R_{ab}\dot{R}_{cb} = 0 \implies \dot{R}_{ab}R_{cb} = -R_{ab}\dot{R}_{cb}$$

Hence on reorderring the terms in the sum on the right, you find

$$\dot{R}_{ab}R_{cb} = -\dot{R}_{cb}R_{ab}$$

and this implies the statement

$$\dot{R}R^T = -(\dot{R}R^T)^T \implies \boxed{\dot{R}R^T \text{ is antisymmetric}}$$

Since $\dot{R}R^T$ is antisymmetric

$$\boxed{(\dot{R}R^T)_{ac} = \dot{R}_{ab}R^T_{bc} = \varepsilon_{acd}\,\omega_d}$$

and hence velocity vector is

$$\underline{v} \equiv \frac{d\underline{r}}{dt} = \frac{dx_a}{dt}\underline{e}_a = \frac{dx'_c}{dt}\underline{e}'_c + \omega\times\underline{r} \equiv v' + \omega\times\underline{r}$$

The acceleration vector is (ω is time independant)

$$\underline{a} \equiv \frac{d^2\underline{r}}{dt^2} = a' + 2\omega\times v' + \omega\times(\omega\times\underline{r})$$

2. Newtownian Dynamics

In this section, we deal with **particles** Particles are an idealization since real objects even if very small have very small a spatial extent.

Particles will be represented by a **point in space** that movies in a trajectory denoted by $\underline{r}(t)$ which is a vector denoting its position at a time t *relative* to a specified **origin**.

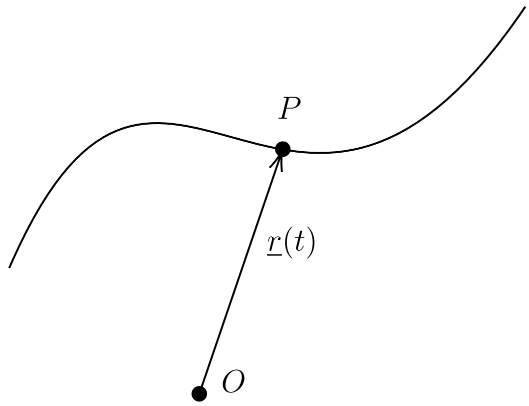
2.1 Basic Kinematics

2.1.1 Position of a particle

Definition 26. A point **particle's positon** at time t on a **trajectory** relative to an **origin** O can be described by a **position vector** *relative* to an origin O

The **position vector** is $\underline{r}$ and can be represented using **basis vectors**.

$$\underline{r}(t) = x\underline{i} + y\underline{j} + z\underline{k} \tag{2.1}$$



Note (Using Einstein Notation to describe position). Using **Einstein's Notation** we can also represented it in the following way:

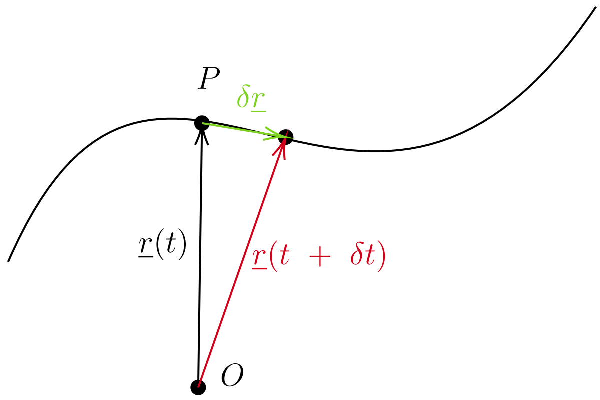
$$\underline{r}(t) = \lambda_a \underline{e_a}$$

2.1.2 Kinematics: Velocity and Acceleration

In position vector of a particle (2.1) $\underline{r}$ assuming that the **Orthonormal bais unit vectors** $\underline{i}, \underline{j}$ and $\underline{k}$ are **constant**, we can write **velocity** and **acceleration** in the following way:

Velocity

Consider the following diagram:



From the diagram above:

$$\delta \underline{r} = \underline{r}(t + \delta t) - \underline{r}(t)$$

Dividing by δt and taking the **limit** as $\delta \to 0$ we get

$$\dot{\underline{r}}(t) = \underline{v}(t) = \lim_{\delta t \to 0} \left(\frac{\underline{r}(t + \delta t) - \underline{r}(t)}{\delta t} \right)$$

Definition 27 (Velocity of a Particle).

$$\dot{\underline{r}} = \frac{d\underline{r}(t)}{dt} = \dot{\underset{\sim}{\lambda}} \underline{e_a} = \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k} \tag{2.2}$$

acceleration

Similarly **acceleration** can be definined in the following way:

Definition 28 (Velocity of a Particle).

$$\ddot{\underline{r}} = \frac{d\dot{\underline{r}}(t)}{dt} = \ddot{\underset{\sim}{\lambda}} \underline{e_a} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k} \tag{2.3}$$

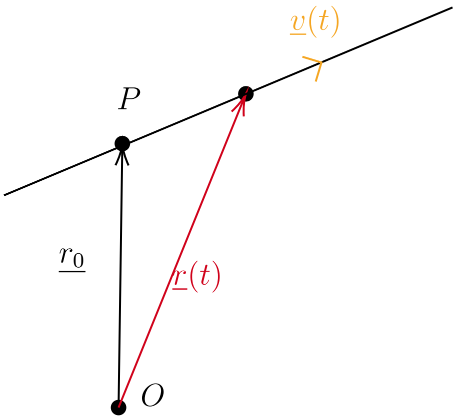
In terms of **limits**

$$\ddot{\underline{r}}(t) = \dot{\underline{v}}(t) = \underline{a}(t) = \lim_{\delta t \to 0} \left(\frac{\underline{v}(t + \delta t) - \underline{v}(t)}{\delta t} \right)$$

2.1.3 Examples of Trajectories

Straight Line Trajectory

Consider the following diagram



Using Vector equation of Lines (1.41), we get the following equation for $\underline{r}(t)$

$$\underline{r}(t) = \underline{r}_0 + t\underline{v} \qquad\qquad \underline{v}, \underline{r}_0 \quad \text{are constants}$$

Then we can find the Velocity (2.2) and Acceleration as (2.3) as

$$\underline{v}(t) = \frac{d\underline{r}(t)}{dt} = \dot{\underline{r}}(t) = \underline{v}$$

$$\underline{a}(t) = \frac{d\dot{\underline{r}}(t)}{dt} = \ddot{\underline{r}}(t) = \underline{0}$$

Parabolic Trajectory

Definition 29 (Parabolic Trajectory). A parabolic trajectory is defined as

$$\underline{r} = \underline{r}_0 + \underline{v}_0 t + \underline{a}_0 \frac{1}{2} t^2 \tag{2.4}$$

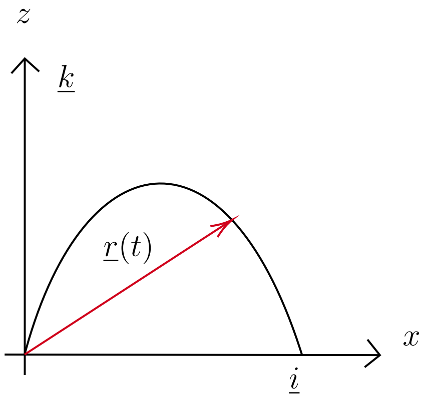
where $\underline{v}_0$ and $\underline{r}_0$ are constants

Acceleration (2.3) and Velocity (2.2) are

- $\underline{v}(t) = \dfrac{d\underline{r}(t)}{dt} = \dot{\underline{r}}(t) = \underline{v}_0 + \underline{a}_0 t$
- $\underline{a}(t) = \dot{\underline{v}}(t) = \dfrac{d}{dt}(\underline{v}_0 + \underline{a}_0 t) = \underline{a}_0$

Example of a Parabolic Trajectory: Consider the following equation:

$$\underline{r}(t) = (\underbrace{u_0\underline{i} + v_0\underline{k}}_{\underline{v}_0})t - \frac{1}{2}gt^2\underline{k}$$



Here, separating the **components** if $\underline{i}$ and $\underline{k}$, we get the following (**scalars**):

$$x(t) = u_0 t \Rightarrow t = \frac{x(t)}{u_0}$$

and substituting for in the value for $z(t)$ (component of $\underline{k}$),

$$z = v_0 t - \frac{1}{2} g t^2$$

$$= \frac{v_0\ x}{u_0} - \frac{1}{2} g \left(\frac{x}{u_0}\right)^2$$

$$= \frac{v_0\ x}{u_0} \left(1 - \frac{1}{2} \frac{gx}{u_0\ v_0}\right)$$

And therefore as we can see, the equation for $z(t)$ is in the form of a **parabola**.

Circular Trajectory

Definition 30 (Circular Trajectory). Consider a particle **trajectory** by the described by the following equations

$$\underline{r}(t) = a(\cos(\omega t)\underline{i} + \sin(\omega t)\underline{j}) \tag{2.5}$$

- $x(t) = a\cos(\omega t)$ i.e. the x-component
- $y(t) = a\sin(\omega t)$ i.e. the y-component

Note.

$$x^2 + y^2 = a^2 \cos^2(\omega t) + a^2 \cos^2(\omega t) \quad \Rightarrow \quad x^2 + y^2 = a^2$$

which is the **equation of a circle** of radius a.

Velocity in a circular trajectory

First calculating the Velocity (2.2),

$$\dot{\underline{r}}(t) = a(-\omega \sin(\omega t)\underline{i} + \omega \cos(\omega t)\underline{j})$$

where:

- $\dot{x}(t) = -a\ \omega \sin(\omega t)$ i.e. the velocity in x-direction

- $\dot{y}(t) = a\ \omega \cos(\omega t)$ i.e. the velocity in y-direction

The **magnitude** of velocity is:

$$| \dot{\underline{r}} |^2 = x^2 + y^2$$

$$= a^2\omega^2 \sin^2(\omega t) + a^2\omega^2 \cos^2(\omega t)$$

$$= a^2\omega^2$$

$$\Rightarrow | \dot{\underline{r}} | = | \underline{v} | = a\omega$$

We can see that velocity has a **constant magnitude**, but is clearly **changing in direction**. The particle is moving in the **anti-clockwise** direction. (This can be verified by *checking any random point*).

Acceleration in a circular trajectory Calculating the Acceleration (2.3),

$$\ddot{\underline{r}}(t) = -a\omega^2(\cos(\omega t)\underline{i} + \sin(\omega t)\underline{j})$$

$$= -\omega^2 \underline{r}$$

So as we can see from the equation $\ddot{\underline{r}}(t) = -\omega^2\underline{r}$, we can see that the acceleration points downwards, i.e. opposite to the direction of $\underline{r}$ i.e. position vector.

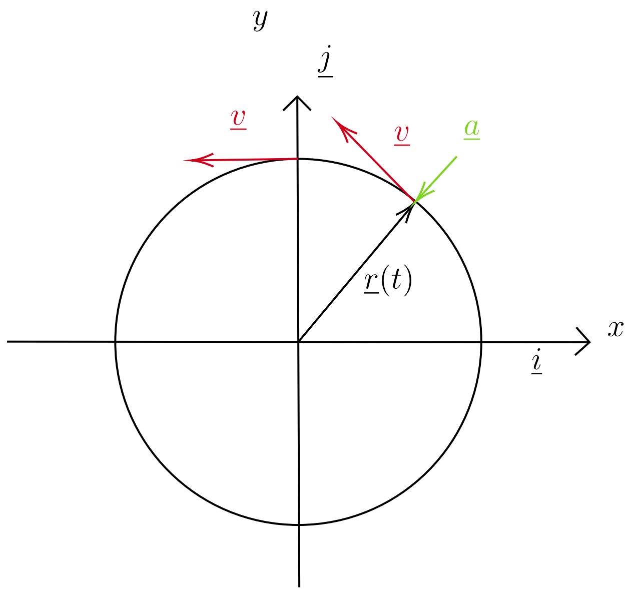


Figure 2.1: Circular Trajectory

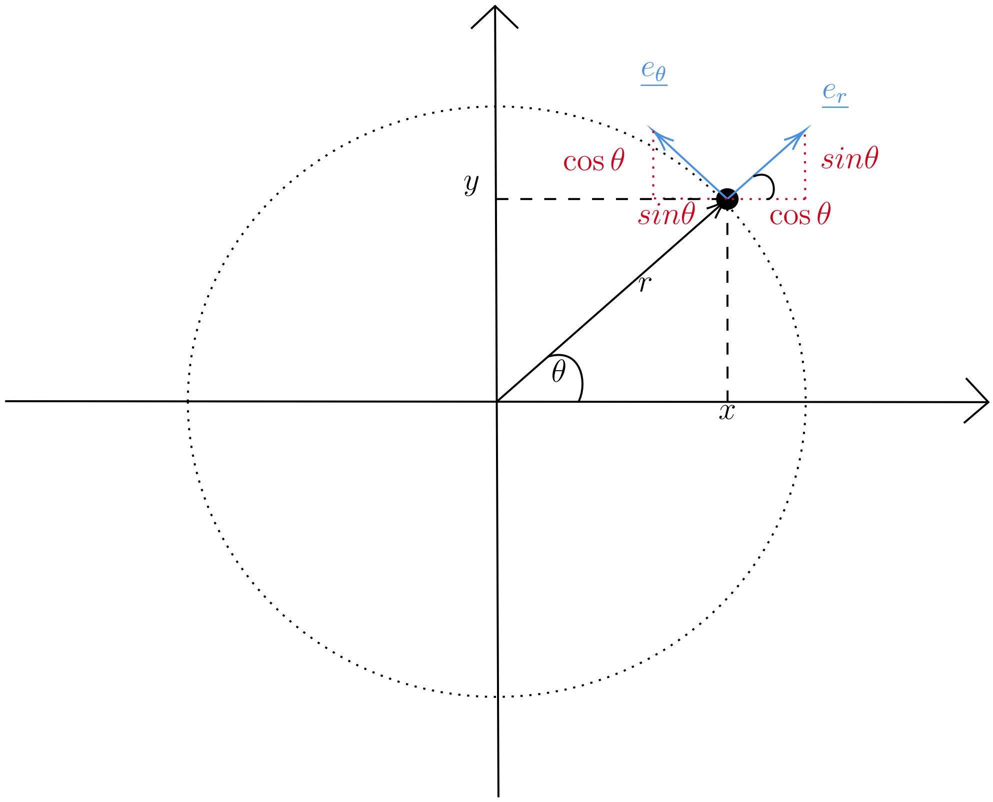
2.2 Motion in Polar Co-ordinates

2.2.1 The polar co-ordinate system

For any **vector** $\underline{x}$ on the $xy-plane$, we can introduce 2 **unit vectors** (not constant)

$$\underline{e_r} \quad , \quad \underline{e_\theta}$$

- $\underline{e_r}$ is the unit vector in the **radial direction**
- $\underline{e_\theta}$ is the unit vector in the **azimuthal direction**



Definition 31 (Relation between Cartesian and Polar Co-ordinates).

$$x = r\cos\theta \quad y = r\sin\theta$$

Where $\mid \underline{r} \mid = r$

Note. Just like basis vectors $\hat{i}$ and $\hat{j}$ in Cartesian co-ordinates, $\underline{e_r}$ and $\underline{e_\theta}$ are orthogonal to each other.

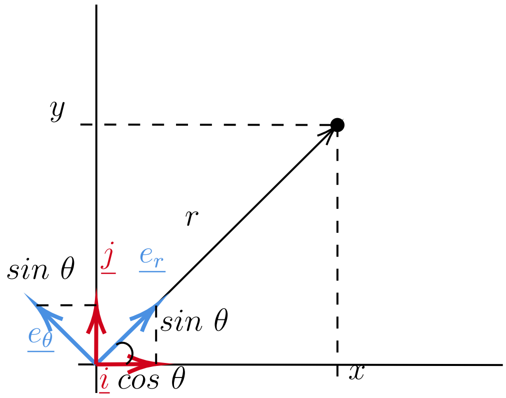
2.2.2 Polar Basis Vectors

Definition 32 (Polar Orthonormal Basis). **Unit Vectors**

$$\underset{\sim}{e_r} \quad \text{and} \quad \underset{\sim}{e_\theta}$$

form the **orthonormal basis** for the **The Polar Co-Ordinate System**.

Furthermore, the unit vectors $\underset{\sim}{e_r}$ and $\underset{\sim}{e_\theta}$ can be represented using **cartesian basis vectors** $\underset{\sim}{i}$ and $\underset{\sim}{j}$



As we can see from the diagram:

$$\underset{\sim}{e_r} = \underset{\sim}{i}\cos\theta + \underset{\sim}{j}\sin\theta$$

and

$$\underset{\sim}{e_\theta} = \pm\sin\theta\ \underset{\sim}{i} + \mp\cos\theta\ \underset{\sim}{j}$$

because $\underset{\sim}{e_r}$ is **orthogonal** to $\underset{\sim}{e_\theta}$. Use the case from the diagram:

$$\underset{\sim}{e_\theta} = \text{-}\underset{\sim}{i}\sin\theta + \underset{\sim}{j}\cos\theta$$

Definition 33 (Polar Basis Using Cartesian).

$$\underset{\sim}{e_r} = \underset{\sim}{i}\cos\theta + \underset{\sim}{j}\sin\theta$$
$$\underset{\sim}{e_\theta} = \text{-}\underset{\sim}{i}\sin\theta + \underset{\sim}{j}\cos\theta$$

Properties of Polar Orthonormal Basis

Theorem 23. Since $\underset{\sim}{e_r}$ and $\underset{\sim}{e_\theta}$ are **orthogonal**,

$$\underset{\sim}{e_r} \cdot \underset{\sim}{e_\theta} = 0$$

(scalar product is 0)

Theorem 24. The cross product (1.30)

$$\underline{e_r} \times \underline{e_\theta} = \underline{\mathsf{K}}$$

Proof.

$$\begin{aligned}
\underline{e_r} \times \underline{e_\theta} &= (\underline{i}\cos\theta + \underline{j}\sin\theta) \times (\underline{i}\sin\theta - \underline{j}\cos\theta) \\
&= (\cos^2\theta + \sin^2\theta)\underline{i} \times \underline{j} \\
&= \underline{k}
\end{aligned}$$

□

2.2.3 Derivatives of Polar Orthonormal Basis Vectors

We assume that the **angle changes with time**, i.e.

$$\theta = \theta\,(t)$$

and therefore the **polar basis vectors** are **NOT CONSTANT**.

First Derivative

First we will compute the first derivatives $\underline{\dot{e_r}}$ and $\underline{\dot{e_\theta}}$.

1. Computing $\underline{\dot{e_r}}$

$$\begin{aligned}
\underline{\dot{e_r}} &= \frac{d}{dt}\Big(\underline{i}\cos\theta + \underline{j}\sin\theta\Big) \\
&= -\dot{\theta}\sin(\theta)\underline{i} + \dot{\theta}\cos(\theta)\underline{j} \\
&= \dot{\theta}(-\sin(\theta)\underline{i} + \cos(\theta)\underline{j}) \\
&= \dot{\theta}\ \underline{e_\theta}
\end{aligned}$$

Definition 34 (First derivative of $\underline{e_r}$).

$$\begin{aligned}
\underline{\dot{e_r}} &= \dot{\theta}\ \underline{e_\theta} \\
&= -\dot{\theta}\sin(\theta)\underline{i} + \dot{\theta}\cos(\theta)\underline{j}
\end{aligned}$$

2. Computing $\dot{\underline{e_\theta}}$

$$\dot{\underline{e_r}} = \frac{d}{dt}\Big(- \sin(\theta)\underline{i} + \cos(\theta)\underline{j}\Big)$$

$$= -\dot{\theta}\cos(\theta)\underline{i} - \dot{\theta}\sin(\theta)\underline{j}$$

$$= -\dot{\theta}(\cos(\theta)\underline{i} + \sin(\theta)\underline{j})$$

$$=\text{-}\dot{\theta}\ \underline{e_r}$$

Definition 35 (First derivative of $\underline{e_r}$).

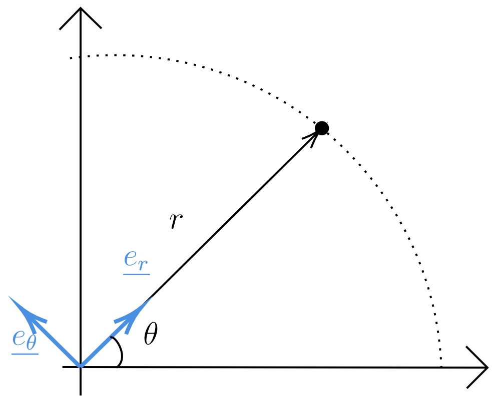
$$\dot{\underline{e_\theta}} = -\dot{\theta}\ \underline{e_r}$$

$$= -\dot{\theta}\cos(\theta)\underline{i} - \dot{\theta}\sin(\theta)\underline{j}$$

2.2.4 Position Vector in Polar Co-ordinates

Definition 36. In polar co-ordinates, **the position vector** i.e. the position of a particle is simply

$$\underline{r} = r\ \underline{e_r}$$



$$\boxed{\underline{r} = r(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}) = r\underline{e_r}}$$

2.2.5 Polar Velocity and Acceleration

Velocity

Computing **velocity** in **polar co-ordinates**:

$$\dot{\underline{r}} = \frac{d}{dt}\Big(r\ \underline{e_r}\Big)$$

$$= \dot{r}\ \underline{e_r} + r\ \dot{\theta}\ \underline{e_\theta} \qquad \text{product rule}$$

Definition 37 (Velocity in Polar co-ordinates).

$$\underline{\dot{r}} = \dot{r}\ \underline{e_r} + r\ \dot{\theta}\ \underline{e_\theta} \tag{2.6}$$

Acceleration

Computing **Acceleration** in **polar co-ordinates**:

$$\begin{aligned}
\underline{\ddot{r}} &= \frac{d}{dt}\Big(\dot{r}(t)\Big) \\
&= \frac{d}{dt}\Big(\dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta}\Big) \\
&= \ddot{r}\underline{e_r} + \dot{r}\dot{e}_r + \dot{r}\dot{\theta}\underline{e_\theta} + r\ddot{\theta}\underline{e_\theta} + r\dot{\theta}\underline{\dot{e}_\theta} \\
&= \ddot{r}\underline{e_r} + \dot{r}\dot{\theta}\underline{e_\theta} + \dot{r}\dot{\theta}\underline{e_\theta} + r\ddot{\theta}\underline{e_\theta} - r\dot{\theta}^2\underline{e_r} \\
&= (\ddot{r} - r\dot{\theta}^2)\underline{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e_\theta}
\end{aligned}$$

Definition 38 (Acceleration in Polar co-ordinates).

$$\ddot{r} = (\ddot{r} - r\dot{\theta}^2)\underline{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e_\theta} \tag{2.7}$$

2.2.6 Cross Product Between Position and Velocity in Polar

We want to compute the cross product 1.31

$$\underline{r} \times \underline{\dot{r}}$$

We can compute it as follows

$$\begin{aligned}
\underline{r} \times \underline{\dot{r}} &= \underline{r} \times (\dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta}) \\
&= (\underline{r} \times \dot{r}\underline{e_r}) + (\underline{r} \times r\dot{\theta}\underline{e_\theta}) \\
&= (r\underline{e_r} \times \dot{r}\underline{e_r}) + (r\underline{er} \times r\dot{\theta}\underline{e_\theta}) \\
&= r\dot{r}(\underline{e_r} \times \underline{e_r}) + r^2\dot{\theta}(\underline{e_r} + \underline{e_\theta}) \\
&= r^2\dot{\theta}\underline{k}
\end{aligned}$$

Note. We have used the **properties of cross product** on cartesian and polar basis vectors

Definition 39 (Cross Product b/w Position and Velocity in Polar).

$$\underline{r} \times \underline{\dot{r}} = r^2 \dot{\theta} \underline{k} \tag{2.8}$$

Note. $\dot{r} \neq \mid \underline{\dot{r}} \mid$ or rather

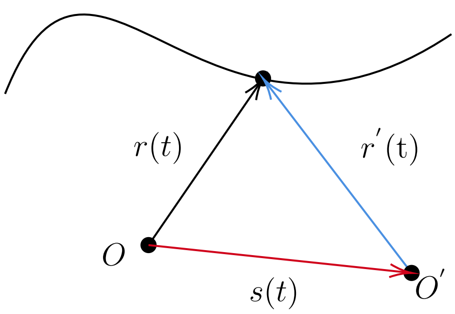
$$\dot{r} = \underline{\dot{r}} \cdot \underline{e_r} = \frac{\underline{r} \cdot \underline{\dot{r}}}{r} \qquad \text{while,} \quad \mid \underline{\dot{r}} \mid = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

2.3 Inertial Frames

The Laws of Physics are the **SAME** in **ALL** inertial frames. Inertial frames are frames of reference which are not accelerating and where Newton's law of inertia holds.

2.3.1 Convertting between Inertial frames

Consider the following diagram:



Here

- The vector $r(t)$ is the position vector of a particle in the inertial frame O/relative to 0.
- The vector $r'(t)$ is the position vector of the same particle in the inertial frame O'/relative to O'.
- The vector $s(t)$ represents the shift between the two frames.

$$\underline{r}(t) = \underline{r}^{'}(t) + \underline{s}(t) \Rightarrow \underline{\dot{r}}(t) = \underline{\dot{r}}^{'}(t) + \underline{\dot{s}}(t)$$

$$\Rightarrow \underline{\ddot{r}}(t) = \underline{\ddot{r}}^{'}(t) + \underline{\ddot{s}}(t)$$

Since we are working with **inertial frames**, they both must have **constant relative velocity**

$$\underline{s}' = \text{constant} \Rightarrow \underline{s}''(t) = 0$$

i.e. the shift acceleration/relative acceleration is zero.

Definition 40 (Inertial Frames). An **inertial frame** is a frame of reference which is not accelerating and where Newton's law of inertia holds.
If we an inertial frame, **the relative/shift acceleration is 0**

$$s''(t) = 0 \tag{$*$}$$

and therefore

$$\ddot{r}(t) = \ddot{r}'(t)$$

2.3.2 Gallilean Transformation

We have seen from ($*$) that to have an inertial frame we had

$$\ddot{\underline{s}}(t) = 0$$

And then we can solve this differential equation with respect to t to get

$$\underline{s}(t) = \underline{a} + \underline{u}t$$

where u is a **constant velocity** and a is a **shift in origin**. This is also known as **Gallilean transformation**.

Definition 41 (Gallilean Transformation). A Gallilean Transformation is when the the **shift vector** $s(t)$ is the following:

$$s(t) = a + ut \tag{2.9}$$

2.4 Newton's Laws of Motion

2.4.1 Inertia

Definition 42 (Law of Inertia). Any body which **isn't** being acted on by an **outside force** stays at rest if it is *initially* at rest, or continues to move at a **constant** velocity if that's what it was doing to begin with. i.e.

Every object will *remain* at **rest** or in **uniform motion** in a **straight line** unless *compelled* to change its state by the action of an **external force**.

2.4.2 Newton's First Law of Motion

Definition 43 (Newton's First Law). Every body **continues** in a state of **rest** or **uniform motion** in a right line unless t is *compelled* to change that state by **forces** impressed on it.

2.4.3 Newton's Second Law of Motion

Definition 44 (Newton's Second Law). The *change* of motion is **proportional** to the **motive force** impressed on it and is *made* in the **direction of the right line** in which that force was impressed.

Newton's second law postulates a *relation* between acceleration (2.3) of the body and the **forces** acting on it. Therefore we can reformulate Newon's second law as follows:

Definition 45 (Newton's Second Law). The **net force** $\underline{F}$ on a body of **constant mass** causes a body to **accelerate**. The acceleration $\ddot{\underline{r}}$ is *in the direction* of $\underline{F}$ **proportional** to the magnitude of the force and **inversely proportional** to the mass of the body:

$$\ddot{\underline{x}} = \frac{\underline{F}}{m}$$

or equivalently

$$\underline{F} = m\ddot{\underline{x}} \tag{2.10}$$

2.4.4 Newton's Third Law of Motion

Definition 46 (Newton's Third Law). To every **action** there is always an **equal and opposite reaction**: or the **mutual actions** of two bodies upon each other are always **equal** and **directed** to **contrary** parts.

2.5 Equation of Motion

Note. Acceleration is **proportional** to the **net force** acting on the body. Therefore, we can write

$$\underline{a} \propto \underline{F}$$

In an **inertial frame**, a particle moves in such a way that its acceleration (2.3) is **proportional** to the sum of all forces acting on it **Newton's Second Law of Motion**

Definition 47 (Equation of Motion). The **equation of motion** of a particle is the **differential equation** that describes the **trajectory** of the particle in space. In an **inertial frame**, the equation of motion is given by

$$\ddot{\underline{r}}(t) = \frac{\underline{F}}{m} \tag{2.11}$$

where $\underline{F}$ is the **net force** acting on the particle and m is the **mass** of the particle.

Also written as

$$\boxed{\underline{F} = m\underline{a} = m\frac{d^2\underline{r}}{dt} = m\ddot{\underline{r}}}$$

2.5.1 Momentum

Definition 48 (Momentum). The **momentum** of a particle is the **product** of its **mass** and **velocity**:

$$\underline{p} = m\underline{v} \tag{2.12}$$

Note. From (2.12), we can see that the **momentum** is a **vector** quantity.

We can generalize the definition of Force usng momentum as follows:

Definition 49 (Newton's Second Law in terms of Momentum). Newton's second law (2.10) can be written in terms of momentum as follows:

$$\underline{F} = \frac{d\underline{p}}{dt} \tag{2.13}$$

2.6 Sample Forces

2.6.1 Gravitational Force

Definition 50 (Gravitational Force). The gravitational force between 2 particles of *mass* m_1 and m_2, situated at $\underline{r_1}$ and $\underline{r_2}$ (i.e. the force felt by particle 1 because of the prescence of particle 2) is given by

$$\underline{F_{12}} = \frac{Gm_1m_2}{\mid \underline{r_1} - \underline{r_2} \mid^2} \frac{\underline{r_2} - \underline{r_1}}{\mid \underline{r_1} - \underline{r_2} \mid}$$

$$\underline{F_{21}} = \frac{Gm_2m_1}{\mid \underline{r_2} - \underline{r_1} \mid^2} \frac{\underline{r_1} - \underline{r_2}}{\mid \underline{r_1} - \underline{r_2} \mid}$$

where the two forces are **equal** and **opposite in direction**:

$$\underline{F_{12}} = -\underline{F_{21}}$$

Note. The vectors:

$$\frac{\underline{r_1} - \underline{r_2}}{\mid \underline{r_1} - \underline{r_2} \mid} \quad \text{and} \quad \frac{\underline{r_2} - \underline{r_1}}{\mid \underline{r_2} - \underline{r_1} \mid}$$

are **unit vectors**. That is they give the *direction* of the gravitational force, and it is in the **direction directed towards each other**.

Gravitational Constant

Definition 51 (Gravitational Constant). The **gravitational constant** G is a **constant** that is used to **quantify** the **attractive force** between two objects with **mass**. It is **approximately** equal to

$$G = 6.674 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$$

Gravitational Force Near the Earth's Surface

Definition 52 (Gravitational Force Near the Earth's Surface). The **gravitational force** near the Earth's surface is given by

$$\underline{F} = m\underline{g} = -mg\underline{k}$$

where m is the **mass** of the object and $\underline{g}$ is the **gravitational acceleration** near the Earth's surface. The **gravitational acceleration** near the Earth's surface is given by

$$g = \frac{Gm_{earth}}{R_{earth}^2} \approx 9.8m/s^2$$

Note. Hence *near* the Earth, **Newton's Equation of Motion** (2.11) becomes:

$$m\ddot{\underline{r}} = -mg\underline{k} \Rightarrow \ddot{\underline{r}} = -g\underline{k}$$

i.e gravitational acceleration is **independent of the mass**.

This differential equation can be solved to give:

$$\underline{r}(t) = \underline{r_0} + t\underline{v_0} - \frac{1}{2}t^2 g\underline{k}$$

2.6.2 Lorrentz Force

Definition 53 (Lorrentz Force). Force on a charged particle in an electromagnetic field $(\underline{E}, \underline{B})$:

$$\underline{F} = q\left(\underline{E} + \dot{\underline{r}} \times \frac{\underline{B}}{c}\right)$$

where q is the charge of the particle, $\underline{E}$ is the electric field, $\underline{B}$ is the magnetic field, and c is the speed of light.

Note. Note mass is additive, charge is not.

Notation. Let $M = \sum_{N}^{i=1}$ be the total mass of the system, and m_i be the mass of the ith particle.

2.7 Energy

2.7.1 Kinetic Energy

Consider Newton's Equation of Motion (2.11):

$$m\ddot{\underline{r}} = \underline{F}$$

We multiply both sides by $\dot{\underline{r}}$ to get:

$$m\ddot{\underline{r}} = \underline{F} \Rightarrow m\dot{\underline{r}} \cdot \ddot{\underline{r}} = \dot{\underline{r}} \cdot \underline{F}$$

$$\Rightarrow m\frac{d}{dt}\left(\frac{1}{2}\dot{\underline{r}} \cdot \dot{\underline{r}}\right) = \underline{F} \cdot \dot{\underline{r}} \qquad \text{(chain rule)}$$

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{m\frac{1}{2} \mid \dot{\underline{r}}^2 \mid}_{\text{Kinetic Energy } K} \Big) = \underline{F} \cdot \dot{\underline{r}}$$

Definition 54 (Kinetic Energy). The **kinetic energy** K of a particle is given by:

$$K = \frac{1}{2}m \mid \dot{\underline{r}} \mid^2 = \frac{1}{2}m \mid \underline{v} \mid^2 \tag{2.14}$$

2.7.2 Work Done

Consider the rate of change of kinetic energy (2.14):

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}m \mid \dot{\underline{r}} \mid^2\right) \Rightarrow \frac{dK}{dt} = \frac{1}{2}m\frac{d}{dt}\left(\mid \dot{\underline{r}} \mid^2\right)$$

$$\Rightarrow \frac{dK}{dt} = m\dot{\underline{r}} \cdot \ddot{\underline{r}}$$

Integrating both sides with respect to time t_1 to t_2 gives:

$$\int_{t_1}^{t_2} m\underline{r} \cdot \ddot{\underline{r}} dt = \int_{t_1}^{t_2} \frac{dK}{dt} dt = K(t_2) - K(t_1)$$

$$= \int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}} \, dt \qquad \text{Here, } \underline{F} = \underline{F}(\underline{r})$$

$$= \int_{P_1}^{P_2} \underline{F} \cdot d\underline{r}$$

Note. P_1 and P_2 are the positions of the particle at times t_1 and t_2 respectively on a trajectory.

The last integral is called a **line integral** and is integrated along the trajectry/curve.

Note

$$\dot{K} = \frac{dK}{dt} = \underline{F} \cdot \dot{\underline{r}}$$

Definition 55 (Work Done). The **work done** W by a force $\underline{F}$ on a particle moving along a trajectory from P_1 to P_2 is given by:

$$W = \int_{P_1}^{P_2} \underline{F} \cdot d\underline{r} = K(t_2) - K(t_1) \tag{2.15}$$

i.e. it is the change in kinetic energy.

2.7.3 Potential Energy

Definition 56 (Conservative Forces). A force $\underline{F}$ is **conservative** if it can be written as the **gradient** of a **scalar function** Φ:

$$\underline{F} = -\underline{\nabla}\Phi$$

where ∇ is the **gradient operator**:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Hence the force is

$$\underline{F} = -\underline{\nabla}\Phi = -\left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)\Phi = -\left(\frac{\partial \Phi}{\partial x}\underline{i} + \frac{\partial \Phi}{\partial y}\underline{j} + \frac{\partial \Phi}{\partial z}\underline{k}\right)$$

Potential Energy and Conservation

Consider the following calculations:

$$\underline{F} = -\underline{\nabla}\Phi \Rightarrow \underline{F} \cdot \dot{\underline{r}} = -\dot{\underline{r}} \cdot \underline{\nabla}\Phi$$

Now by the definition of Kinetic Energy (2.14)

$\underline{F} \cdot \dot{\underline{r}} = dK/dt = \dot{K}$, we get the following:

$$\frac{dK}{dt} = -\dot{\underline{r}} \cdot \underline{\nabla}\Phi \Rightarrow \frac{dK}{dt} = -\dot{\underline{r}} \cdot \underline{\nabla}\Phi(\underline{r})$$

$$\Rightarrow \frac{dK}{dt} = -\frac{d\Phi}{dt} \qquad \text{chain rule}$$

$$\Rightarrow \frac{d}{dt}\Big(K + \Phi\Big) = 0$$

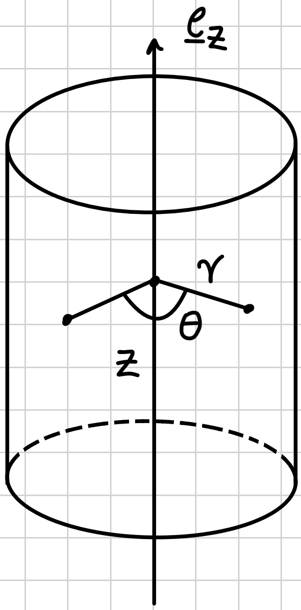
And therefore Energy is a **conserved quantity**.

More notes on cylindrical polars

We know from vector calculus,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$\Rightarrow \boxed{\underline{r} = r(\cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2) + z\underline{e}_3}$$



As we saw above,

$$\underline{e}_r = \cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2$$

$$\Rightarrow \boxed{\underline{r} = r\underline{e}_r + z\underline{e}_z}$$

The gradient operator on polar is

$$\boxed{\begin{aligned} \underline{\nabla} &= \left(\frac{\partial}{\partial x}\underline{e}_1 + \frac{\partial}{\partial y}\underline{e}_2 + \frac{\partial}{\partial y}\underline{e}_3\right) \\ \underline{\nabla} &= \left(\underline{e}_r\frac{\partial}{\partial r} + \underline{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \underline{e}_z\frac{\partial}{\partial z}\right) \end{aligned}}$$

and $\boxed{\underline{e}_r \text{ and } \underline{e}_\theta \text{ and } \underline{e}_z \text{ orthogonal}}$

and $\boxed{\dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta + \dot{z}\underline{e}_z}$

Remember

$\underline{e}_r$, $\underline{e}_\theta$ NOT CONSTANT

$\underline{e}_z = \underline{e}_3$ constant

Definition 57 (Conservation of Energy). Energy is a **constant of motion**

$$\dot{E} = \frac{d}{dt}\Big(K + \Phi\Big) = 0$$

Therefore

$$E = K + \Phi = \text{CONSTANT}$$

Definition 58 (Potential Energy). The **potential energy** Φ is given by:

$$\Phi = -\int_{P_1}^{P_2} \underline{F} \cdot d\underline{r} \tag{2.16}$$

2.8 Example Conservative Forces

2.8.1 Gravitational Force Near the Earth's Surface

As shown in the previous section, the gravitational force near the Earth's surface is given by

$$\underline{F} = m\underline{g} = -mg\underline{k}$$

where m is the **mass** of the object and $\underline{g}$ is the **gravitational acceleration** near the Earth's surface.
And therefore we can derive the following:

$$-mg\underline{k} = \Big(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\Big)(-mgz)$$

$$= -\underline{\nabla}(mgz)$$

Therefore we can define the following:

Definition 59 (Gravitational Potential Energy Near the Earth's Surface). The **gravitational potential energy** Φ is given by:

$$\Phi = mgz \tag{2.17}$$

Example (Calculating Velocity). A particle is dropped from rest at a height $z = h$. Calculate the velocity

Solution:
We know that the particle is dropped from rest, therefore $\underline{v} = 0$ at $t = 0$ at height $z = h$ Therefore.

$$E = K + \Phi = \frac{1}{2}m \mid \underline{0} \mid^2 + mgh = mgh$$

At height $z = 0$, height is 0 so

$$E = K + \Phi = \frac{1}{2}m \mid \dot{\underline{r}} \mid^2 + mg0 = \frac{1}{2}m \mid \dot{\underline{r}} \mid^2$$

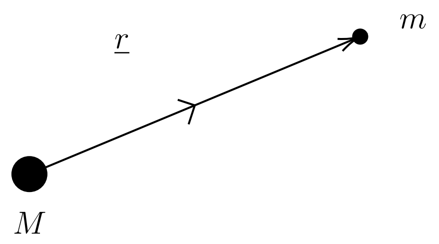
Since **Energy is a constant**, we get that:

$$mgh = \frac{1}{2}m \mid \dot{\underline{r}} \mid^2 \Rightarrow 2gh = \mid \dot{\underline{r}} \mid^2$$

$$\Rightarrow \mid \dot{\underline{r}} \mid = \sqrt{2gh}$$

2.8.2 Gravitational Potential Energy Away from the Earth's Surface

Consider the following diagram:



The gravitational potential energy is given by:

$$\underline{F} = m\ddot{\underline{r}} = -\frac{mMG}{\mid \underline{r} \mid^2}\frac{\underline{r}}{\mid \underline{r} \mid}$$

$$= -\underline{\nabla}\Big(-\frac{mMG}{\mid \underline{r} \mid}\Big)$$

Definition 60 (Gravitational Potential Away From Earths Surface). The **gravitational potential** Φ is given by:

$$\Phi = -\frac{mMG}{\mid \underline{r} \mid} \tag{2.18}$$

2.9 Angular Momentum

Definition 61 (Angular Momentum). The **angular momentum** $\underline{J}$ of a particle is given by:

$$\underline{J} = \underline{r} \times \underline{p} = m\underline{r} \times \dot{\underline{r}} \tag{2.19}$$

where $\underline{r}$ is the position vector of the particle, and $\underline{p}$ is the momentum (2.12) of the particle.

2.9.1 Moment of a Force

From the following calculation:

$$\begin{aligned}
\underline{\dot{J}} &= \frac{d}{dt}\Big(m\underline{r} \times \underline{\dot{r}}\Big) \\
&= m\underline{\dot{r}} \times \underline{\dot{r}} + m\underline{r} \times \underline{\ddot{r}} && \text{product rule} \\
&= 0 + m\underline{r} \times \underline{\ddot{r}} && \text{properties of cross product} \\
&= m\underline{r} \times \underline{\ddot{r}} \\
&= \underline{r} \times \underline{F} && \text{Newton's Equation of Motion (2.11)} \\
&\equiv \underline{M}
\end{aligned}$$

Definition 62 (Moment of a Force). The **moment of a force** $\underline{M}$ of a particle is defined to the **rate of change of angular momentum** (2.19) is given by:

$$\underline{M} = \underline{r} \times \underline{F} \tag{2.20}$$

where $\underline{r}$ is the position vector of the particle, and $\underline{F}$ is the **force** on the particle.

Note. $\underline{M}$ is also called the **torque** of the force $\underline{F}$.

2.9.2 Conservation of Angular Momentum

Consider a particle moving under the influence of a force $\underline{F}$ directed towards or away from the origin.

$$\underline{F} = f(\underline{r})\underline{r}$$

where $f(\underline{r})$ is a scalar. Hence calculating the moment of the force $\underline{F}$:

$$\begin{aligned}
\underline{\dot{J}} = \underline{r} \times \underline{F} &= f(\underline{r})\underline{r} \times \underline{r} \\
&= 0 \\
\Rightarrow \underline{\dot{J}} &= 0
\end{aligned}$$

Hence angular momentum is a **conserved quantity**.

Definition 63 (Conservation of Angular Momentum). If a force $\underline{F}$ is **proportional** to $\underline{r}$, i.e.

$$\underline{F} = f(\underline{r})\underline{\boldsymbol{r}}$$

then angular momentum is conserved:

$$\underline{\dot{J}} = 0$$

2.10 Collection of particles

2.10.1 Total Force in a collection of particles

In a discrete system of N **particles**, of mass m_i and positions $\underline{r_i}(t)$, *relative* to a chosen origin O.
The **particle** i experiences **two** types of forces:

1. **External forces** $\underline{F_i}^{ext}$ maybe due to external fields (e.g. gravitational, electric, magnetic, etc.) where $i \in \{1 \dots N\}$
2. **Inter-Particle forces** $\underline{F_{ij}}$ due to the presence of other particles.

Therefore from Newton's second law, the equation of motion for particle (2.10) i is:

Definition 64 (Force on Particle i). For particles i where $i \in \{1 \dots N\}$, the force on particle i is given by:

$$m_i \underline{\ddot{r}_i} = \underline{F_i}^{ext} + \sum_{j=1}^{N} \underline{F_{ij}} \tag{2.21}$$

where $\underline{F_{ij}}$ is the force on particle i due to particle j.

Note. Particle i does not feel a force from itself, i.e. $\underline{F_{ii}} = 0$.

Due to **Newton's third law**, the force on particle j due to particle i is **equal and opposite** to the force on particle i due to particle j, i.e.

$$\underline{F_{ij}} = -\underline{F_{ji}}$$

.

Hence summing on index i in (2.21) gives:

$$\sum_{i=1}^{N} m\underline{\ddot{r}_i} = \underbrace{\sum_{i,j=1}^{N} \underline{F_{ij}}}_{0 \text{ because } F_{ij}=-F_{ji}} + \sum_{i=1}^{N} \underline{F_i}^{\text{ext}}$$

$$= 0 + \sum_{i=1}^{N} \underline{F_i}^{\text{ext}}$$

Definition 65 (Total Force). The **total force** on the system is given by:

$$\underline{F}^{\text{ext}} = \sum_{i=1}^{N} \underline{F_i}^{\text{ext}} \tag{2.22}$$

i.e. the sum of all external forces on the system.

2.10.2 Center of Mass

Definition 66 (Center of Mass). In a discrete system of N particles with **masses** m_i and **position vectors** $\underline{r}_i$, relative to a fixed origin O, the **center of mass** is defined as

$$\underline{R} = \frac{\sum_{i=1}^{N} m_i \underline{r_i}}{\sum_{i=1}^{N} m_i} \tag{2.23}$$

Note. The denominator of (2.23) is the **total mass** of the system which will be denoted by

$$M = \sum_{i=1}^{N} m_i \tag{2.24}$$

and hence

$$\underline{R} = \frac{\sum_{i=1}^{N} m_i \underline{r_i}}{M} \tag{2.25}$$

Definition 67 (Total External Force using Center of Mass). The **total external force** acting on the system is defined as

$$M\ddot{\underline{R}} = \underline{F_{\text{total}}^{(e)}} \tag{2.26}$$

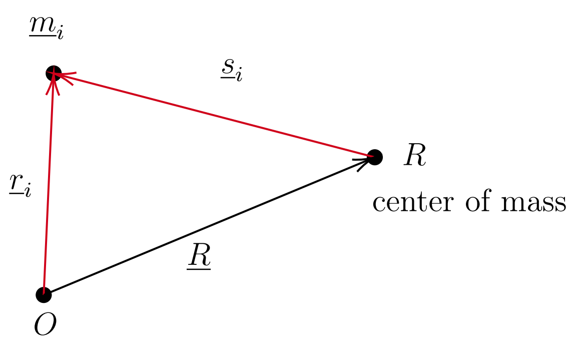
Note. If **total external force is zero**, i.e. $\underline{F_{\text{total}}^{(e)}} = 0$, then $\ddot{\underline{R}} = 0$ and therefore $\dot{\underline{R}}$ is contant. Hence the center of mass **moves with constant velocity**.

2.10.3 Total Kinetic Energy in a Collection of Particles

Definition 68 (Total Kinetic Energy). In a discrete system of N particles with **mass** m_i and position vector $\underline{r_i}\,(t)$, the **total kinetic energy** of a collection of particles is defined as

$$K_{tot} = \sum_{i=1}^{N} \frac{1}{2} m_i \mid \dot{\underline{r}}_i \mid^2 \tag{2.27}$$

Consider the following diagram (R is the center of mass 2.23):



Set $\underline{s}_i = \underline{r}_i - \underline{R}$, then

$$K_{tot} = \frac{1}{2}\sum_{i=1}^{N} m_i \mid \dot{\underline{r}_i} \mid^2 \Rightarrow K_{tot} = \frac{1}{2}\sum_{i=1}^{N} m_i \mid \dot{\underline{R}} + \dot{\underline{s}_i} \mid^2$$

$$\Rightarrow K_{tot} = \frac{1}{2}\sum_{i=1}^{N} \left[m_i \mid \dot{\underline{R}} \mid^2 + m_i 2\dot{\underline{R}} \cdot \dot{\underline{s}_i} + m_i \mid \dot{\underline{s}_i} \mid^2 \right] \tag{1}$$

Note. From the definition of the **center of mass** (2.23), we have

$$M\underline{R} = \sum_{i=1}^{N} m_i \underline{r_i} = \sum_{i=1}^{N} m_i (\underline{R} + \underline{s}_i)$$

$$= M\underline{R} + \sum_{i=1}^{N} m_i \underline{s_i}$$

and therefore, we get

$$\sum_{i=1}^{N} m_i \underline{s_i} = 0$$

and therefore, (1) becomes

$$K_{tot} = \frac{1}{2} M \mid \dot{\underline{R}} \mid^2 + \frac{1}{2} m_i \mid \dot{\underline{s}_i} \mid^2 \qquad \text{(summation convention)}$$

where $M = \sum_{i=1}^{N} m_i$

Definition 69 (Total Kinetic Energy v2). In a discrete system of N particles with **mass** m_i and position vector $\underline{r_i}(t)$, the **total kinetic energy** of a collection of particles is defined as

$$K_{tot} = \frac{1}{2} M \mid \dot{\underline{R}} \mid^2 + \frac{1}{2}\sum_{i=1}^{N} m_i \mid \dot{\underline{s}_i} \mid^2 \tag{2.28}$$

2.10.4 Total Angular Momentum

Consider the following calculations:

$$\underline{J}_{tot} = \sum_{i=1}^{N} m_i \underline{r_i} \times \dot{\underline{r_i}} \qquad \left(\dot{\underline{J}}_{tot} = \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i \right)$$

$$= \sum_{i=1}^{N} m_i (\underline{R} + \underline{s_i}) \times (\dot{\underline{R}} + \dot{\underline{s_i}}) \qquad \text{since } \underline{r_i} = \underline{s_i} + \underline{R}$$

$$= M\underline{R} \times \dot{\underline{R}} + \frac{1}{2} \sum_{i=1}^{N} m_i \underline{s_i} \times \dot{\underline{s_i}}$$

Definition 70 (Total Angular Momentum). In a discrete system of N particles with **masses** m_i and **position vectors** $\underline{r}_i$, relative to a fixed origin O, the **total angular momentum** of a collection of particles is defined as

$$\underline{J}_{tot} = M\underline{R} \times \dot{\underline{R}} + \frac{1}{2} \sum_{i=1}^{N} m_i \underline{s_i} \times \dot{\underline{s_i}} \tag{2.29}$$

2.10.5 N-body Gravitational System

For N-body Gravitational System with no external forces, moving under mutual **gravitational forces**, we calculate the **rate of change of Kinetic Energy** (2.14) of the system.

$$\dot{K}_{tot} = \frac{d}{dt} \sum_{i=1}^{N} \frac{1}{2} m_i \mid \dot{\underline{r}}_i \mid^2$$

$$= \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \ddot{\underline{r}}_i$$

Note. For a gravitational system

$$m_i \ddot{\underline{r}}_i = \sum_{\substack{j=1 \\ \neq i}} \frac{G m_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid^2} \frac{\underline{r_j} - \underline{r_i}}{\mid \underline{r_i} - \underline{r_j} \mid}$$

We can write the **rate of change of Kinetic Energy** (2.14) of the system as

$$\dot{K}_{tot} = \sum_{i=1}^{N} \dot{\underline{r}}_i \cdot \sum_{\substack{j=1 \\ \neq i}}^{N} \frac{Gm_i m_j}{\mid \underline{r}_i - \underline{r}_j \mid^2} \frac{\underline{r}_j - \underline{r}_i}{\mid \underline{r}_i - \underline{r}_j \mid}$$

$$= \sum_{\substack{j,i=1 \\ i \neq j}}^{N} \frac{Gm_i m_j}{\mid \underline{r}_i - \underline{r}_j \mid^3} \quad \dot{\underline{r}}_i \cdot (\underline{r}_j - \underline{r}_i)$$

$$= \frac{1}{2} \sum_{\substack{j,i=1 \\ i \neq j}}^{N} \frac{Gm_i m_j}{\mid \underline{r}_i - \underline{r}_j \mid^3} (\dot{\underline{r}}_i - \dot{\underline{r}}_j) \cdot (\underline{r}_j - \underline{r}_i) \tag{$*$}$$

Remark.

$$\frac{d}{dt} \mid \underline{p} \mid^2 = 2 \mid \underline{p} \mid \frac{d \mid \underline{p} \mid}{dt}$$

and

$$\frac{d}{dt} \mid \underline{p} \mid^2 = \frac{d}{dt} (\underline{p} \cdot \underline{p}) = 2\underline{p} \cdot \dot{\underline{p}}$$

And therefore we get

$$2 \mid \underline{p} \mid \frac{d \mid \underline{p} \mid}{dt} = 2\underline{p} \cdot \dot{\underline{p}}$$

$$\Rightarrow \frac{d \mid \underline{p} \mid}{dt} = \frac{\underline{p} \cdot \dot{\underline{p}}}{\mid \underline{p} \mid}$$

Note.

$$\frac{d}{dt} \frac{1}{\mid \underline{r}_i - \underline{r}_j \mid} = - \frac{1}{\mid \underline{r}_i - \underline{r}_j \mid^2} \frac{d}{dt} \mid \underline{r}_i - \underline{r}_j \mid$$

$$= - \frac{1}{\mid \underline{r}_i - \underline{r}_j \mid^3} (\underline{r}_i - \underline{r}_j) \cdot (\dot{\underline{r}}_i - \dot{\underline{r}}_j)$$

$$\boxed{\frac{d}{dt}\left(\frac{1}{\mid \underline{u} \mid}\right) = -\frac{1}{\mid \underline{u} \mid^2} \frac{\underline{u} \cdot \dot{\underline{u}}}{\mid \underline{u} \mid}}$$

Therefore equation ($*$) becomes

$$\dot{K}_{tot} = \frac{1}{2} \sum_{\substack{j,i=1 \\ i \neq j}}^{N} \frac{d}{dt} \frac{Gm_i m_j}{\mid \underline{r}_i - \underline{r}_j \mid}$$

$$= \sum_{i<j}^{N} \frac{d}{dt} \frac{Gm_i m_j}{\mid \underline{r}_i - \underline{r}_j \mid} \tag{$**$}$$

Since $i < j$ is already **half the number** of terms in the sum.

And using the **summation properties for derivatives** (**) becomes

$$\dot{K}_{tot} = \sum_{i<j}^{N} \frac{d}{dt} \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid}$$

$$= \frac{d}{dt} \sum_{i<j}^{N} \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid}$$

Therefore we get

$$\dot{K}_{tot} - \frac{d}{dt} \sum_{i<j}^{N} \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid} = 0 \Rightarrow \frac{dK_{tot}}{dt} - \frac{d}{dt} \sum_{i<j}^{N} \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid} = 0$$

$$\Rightarrow \frac{d}{dt} \left(K_{tot} - \sum_{i<j}^{N} \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid} \right) = 0$$

That is a **Total Energy is conserved**

Definition 71 (Total Energy in N-body system). The total energy E in an N-body gravity system is

$$E = K_{tot} - \sum_{i<j}^{N} \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid} \tag{2.30}$$

or using (2.28), we get

$$E = \frac{1}{2} M \mid \dot{\underline{R}} \mid^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \mid \dot{\underline{s}_i} \mid^2 - \sum_{i<j}^{N} \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid} \tag{2.31}$$

Definition 72 (Potential Energy in N-body gravitational system). The term

$$-\sum_{i<j}^{N} \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid} \tag{2.32}$$

is the **total gravitational potential energy** expressed as a *sum over all pairs of particles.*

2.10.6 A Virial Theorem

Define the following:

$$D = \frac{1}{2}\sum_{i-1}^{N} m_i \mid \underline{r_i}^2 \mid$$

Then we get the following for the derivatives

$$\dot{D} = \sum_{i=1}^{N} m_i \underline{r_i} \cdot \underline{\dot{r_i}} \qquad \text{chain rule}$$

and the second derivative is (from the product rule)

$$\ddot{D} = \sum_{i=1}^{n} m_i \underline{\dot{r_i}} \cdot \underline{\dot{r_i}} + \sum_{i=1}^{n} m_i \underline{\dot{r_i}} \cdot \underline{\ddot{r_i}} \tag{$*$}$$

Then therefore we can rewrite the equation ($*$) using definition of (2.14)

$$\ddot{D} = 2K_{tot} + \sum_{i=1}^{n} m_i \underline{\dot{r_i}} \cdot \underline{\ddot{r_i}}$$

Virial Theorem on Gravity

Gravitational Force of Attraction is defined as

$$m\underline{\ddot{r_i}} = \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid^2} \frac{\underline{r_j} - \underline{r_i}}{\mid \underline{r_i} - \underline{r_i} \mid}$$

And therefore substituting this into the **second derivative** $\ddot{D}$ we get the following:

$$\begin{aligned}
\ddot{D} &= 2K_{tot} + \sum_{i=1}^{N} \underline{r_i} \cdot \sum_{i \neq j} \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid} \frac{\underline{r_j} - \underline{r_i}}{\mid \underline{r_i} - \underline{r_j} \mid} \\
&= 2K_{tot} + \frac{1}{2}\sum_{i \neq j} (\underline{r_i} - \underline{r_j}) \cdot \frac{Gm_i m_j}{\mid \underline{r_i} - \underline{r_j} \mid^3} (\underline{r_i} - \underline{r_j}) \\
&= 2K_{tot} + \Phi \\
&= K_{tot} + K_{tot} + \Phi
\end{aligned}$$

$$\Rightarrow \ddot{D} = K_{tot} + \underbrace{E}_{\text{conserved}}$$

Define **average Kinetic Energy**

Definition 73 (Average Kinetic Energy).

$$< K_{tot} >= \frac{1}{\tau}\int_0^{\tau} K_{tot}dt \tag{2.33}$$

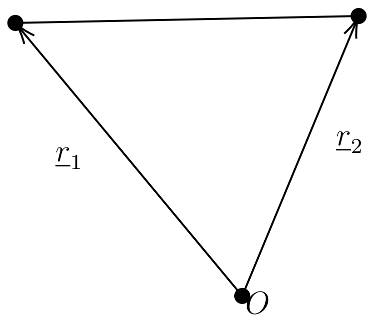
Suppose the quantity $\underline{R}$ does not change, we find that

$$\underline{E} = - < K_{tot} > \quad \text{or} \quad 2 < K_{tot} >= - < V_{tot} >$$

This fact was the basis of an analysis of the Coma cluster of galaxies by Zwicky ('On the Masses of Nebulae and of Clusters of Nebulae', F Zwicky, Astrophysical Journal, vol. 86 (1937) 217), which demonstrated that there should be some kind of 'dark matter' to account for observation. So far, 'dark matter' has not been identified directly though there are other, independent, indications that it should exist and many theories as to what it might be. (For example, see 'Particle dark matter: evidence, candidates and constraints', G Bertone, D Hooper, J Silk, Physics Reports 405 (2005) 279).

2.11 Two-Body Gravitational System

Consider the following diagram



We have 2 **equations of motion** (2.10)

$$m\ddot{\underline{r_1}} = Gm_1m_2\frac{\underline{r_2} - \underline{r_1}}{\mid \underline{r_2} - \underline{r_1} \mid^3} \tag{G1}$$

$$m\ddot{\underline{r_2}} = Gm_1m_2\frac{\underline{r_1} - \underline{r_2}}{\mid \underline{r_2} - \underline{r_1} \mid^3} \tag{G2}$$

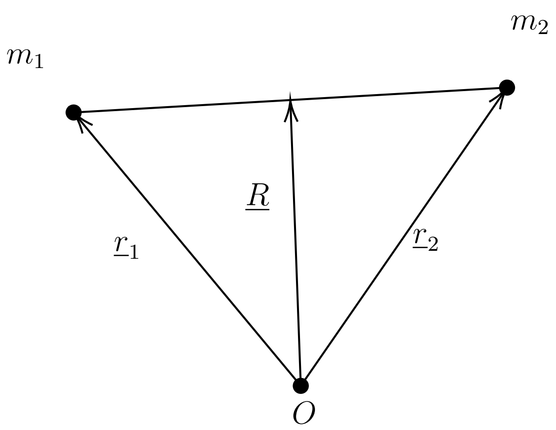
Since the direction vectors are in opposite direction:

$$m_1\ddot{\underline{r_1}} + m_2\ddot{\underline{r_2}} = 0 \Rightarrow (m_1 + m_2)\ddot{\underline{R}} = 0$$

$$\Rightarrow \quad M\ddot{\underline{R}} = 0$$

where $M = m_1 + m_2$, $M\underline{R} = m_1\underline{r}_1 + m_2\underline{r}_2$

Consider the following diagram



Put $\underline{r_1} = \underline{R} + \underline{s_1}$ and $\underline{r_1} = \underline{R} + \underline{s_2}$

$$m_1\ddot{\underline{s_1}} = Gm_1m_2\frac{\underline{s_2} - \underline{s_1}}{\mid \underline{s_2} - \underline{s_1} \mid^3} \qquad m_2\ddot{\underline{s_2}} = Gm_2m_1\frac{\underline{s_1} - \underline{s_2}}{\mid \underline{s_1} - \underline{s_2} \mid^3}$$

and put $\underline{r} = \underline{r_1} - \underline{r_2} = \underline{s_1} - \underline{s_2}$

We get a **second order differential equation**

$$\ddot{\underline{r}} = -G(m_1 + m_2)\frac{\underline{r}}{\mid \underline{r} \mid^3} = -\frac{GM\underline{r}}{\mid \underline{r} \mid^3}$$

Note.

$$\underline{s_1} = \frac{m_2\underline{r}}{m_1 + m_2} \qquad \underline{s_1} = \frac{-m_2\underline{r}}{m_1 + m_2}$$

From an earlier result:

$$E = \frac{1}{2}m_1 \mid \dot{\underline{r_1}} \mid^2 + \frac{1}{2}m_2 \mid \dot{\underline{r_2}} \mid^2 - \frac{Gm_1m_2}{\mid \underline{r_1} - \underline{r_2} \mid}$$

$$= \frac{1}{2}m_1 \mid \dot{\underline{R}} + \dot{\underline{s_1}} \mid^2 + \frac{1}{2}m_2 \mid \dot{\underline{R}} + \dot{\underline{s_2}} \mid^2 - \frac{Gm_1m_2}{\mid \underline{r_1} - \underline{r_2} \mid}$$

$$= \frac{1}{2}m_1 \left| \dot{\underline{R}} + \frac{m_2\dot{\underline{r}}}{m_1 + m_2} \right|^2 + \frac{1}{2}m_2 \left| \dot{\underline{R}} + \frac{m_1\dot{\underline{r}}}{m_1 + m_2} \right|^2 - \frac{Gm_1m_2}{\mid \underline{r_1} - \underline{r_2} \mid}$$

$$= \underbrace{\frac{1}{2}(m_1 + m_2)\left|\dot{\underline{R}}\right|^2}_{(1)} + \underbrace{\frac{1}{2}\frac{m_1m_2}{m_1 + m_2}|\dot{\underline{r}}|^2}_{(2)} - \frac{Gm_1m_2}{|\underline{r}|} \qquad (*G)$$

$\ddot{R} = 0$

In $(*G)$, we have the following:

1. (1) is **conserved** since center of mass acceleration is constant and hence velocity is constant and therefore the **Energy associated with COM** is constant
2. (2) is nothing but Kinetic Energy + Potential Energy which is **constant** and hence **conserved**.

and therefore **Energy in a 2 body gravitational system is conserved**

2.11.1 Angular Momentum in a 2-body gravitational system

By the definition of **Angular Momentum** (2.19)

$$\begin{aligned}
\underline{J} &= m_1 \underline{r_1} \times \underline{\dot{r_2}} + m_2 \underline{r_2} \times \underline{\dot{r_2}} \\
&= m_1(\underline{R} + \underline{s_1}) \times (\underline{\dot{R}} + \underline{\dot{s_1}}) + m_2(\underline{R} + \underline{s_2}) \times (\underline{\dot{R}} + \underline{\dot{s_2}}) \\
&= (m_1 + m_2)(\underline{R} \times \underline{\dot{R}}) + m_1 \underline{s_1} \times \underline{\dot{s_1}} + m_2 \underline{s_2} \times \underline{\dot{s_2}} \\
* \quad &= (m_1 + m_2)\underline{R} \times \underline{\dot{R}} + \frac{m_1 m_2}{m_1 + m_2} \underline{r} \times \underline{\dot{r}} \qquad \text{by substituting } \underline{s_1} \text{ and } \underline{s_2}
\end{aligned}$$

$* \ \ddot{\underline{R}} = 0 \Rightarrow \underline{R} \times \underline{\dot{R}} = 0$ as they lie along same line

and again these are both separately conserved as shown before in **conservation of angular momentum**, since we the force is proportional to $\underline{r}$, i.e.

$$\underline{\ddot{r}} \propto \underline{r}$$

angular momentum is conserved

> **Remark.** In the end, we do not need to care about the **center of mass** (2.23) motion since as it is a **constant velocity motion and hence it is conserved** and has no contribution to angular momentum and energy of the system.

2.11.2 Reduced set of equations ignoring $\underline{R}$

Ignoring $\underline{R}$, the system of equations gets reduced to

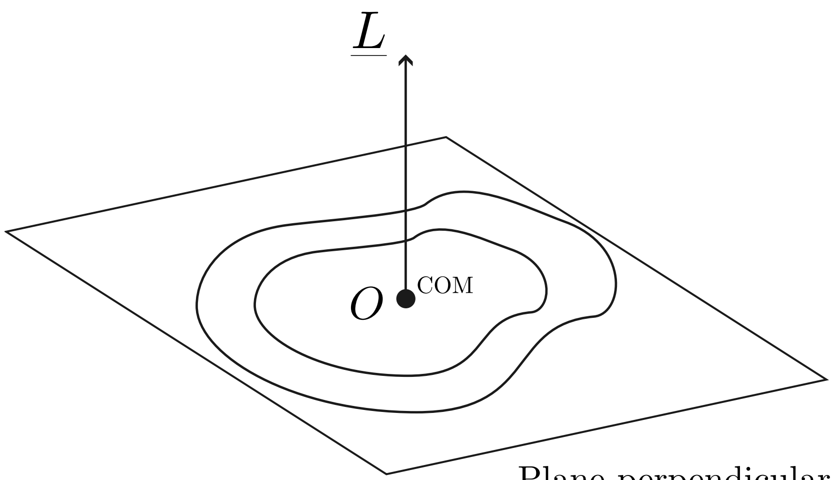
$$\underline{\ddot{r}} = -\frac{G(m_1 + m_2)}{|\underline{r}|^3} \underline{r}$$

$$\varepsilon = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\underline{\dot{r}}|^2 - \frac{G m_1 m_2}{|\underline{r}|}$$

$$\underline{L} = \frac{m_1 m_2}{m_1 + m_2} \underline{r} \times \underline{\dot{r}}$$

Here both $\underline{L}$ and ε are constants.

> **Remark.** Note that $\underline{L}$ is **conserved** and **perpendicular** to $\underline{r}$ and $\dot{\underline{r}}$ and therefore we can use polar co-ordinates.



2.11.3 Solving in Polar Co-ordinates

Let

$$M = (m_1 + m_2) \qquad \text{and} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Then the required equations become

$$\ddot{\underline{r}} = -\frac{GM\underline{r}}{|\underline{r}|^3}$$

$$\varepsilon = \frac{1}{2}\mu|\underline{r}|^2 - \frac{Gm_1 m_2}{|\underline{r}|}$$

$$\underline{L} = \mu\underline{r} \times \dot{\underline{r}}$$

> **Note.**
> $$\underline{L} \cdot \underline{r} = 0 \;\text{ and }\; \underline{L} \cdot \dot{\underline{r}} = 0$$
> So the motion is orthogonal to $\underline{L}$

Solving in **polar co-ordinates** to describe $\underline{r}$,

$$\underline{r} = |\underline{r}|\underline{e_r} \equiv r\underline{e_r}$$

Then as seen before the derivatives of $\underline{r}$ are:

$$\dot{\underline{r}} = \dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta}$$

$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{e_r} + (2r\dot{\theta} + r\ddot{\theta})\underline{e_\theta}$$

Therefore the **angular momentum** can be seen as

$$\begin{aligned}
\underline{L} &= \mu \underline{r} \times \underline{\dot{r}} \\
&= \mu r \underline{e_r} \times (\dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta}) \\
&= \mu r^2 \dot{\theta} \underline{e_r} \times \underline{e_\theta}
\end{aligned}$$

And therefore calculating the magnitude of **Angular Momentum** (2.19),

$$\begin{aligned}
|\underline{L}| &= \mu r^2 \dot{\theta} |\underline{e_r} \times \underline{e_\theta}| \\
&= \mu r^2 \dot{\theta} |\underline{e_r}||\underline{e_\theta}| \sin\frac{\pi}{2} && \text{orthonormal basis vectors} \\
&= \mu r^2 \dot{\theta}
\end{aligned}$$

$$\Rightarrow |\underline{L}| = \text{ CONSTANT } = \mu r^2 \dot{\theta}$$

It is **convention** to represent $\boxed{r^2\dot{\theta} = h}$ and hence we get:

$$\mu r^2 \dot{\theta} = \mu h \Rightarrow \boxed{r^2 \dot{\theta} = h}$$

Furthermore, calculating the magnitude of **velocity** (2.2),

$$|\underline{\dot{r}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + \frac{h^2}{r^2}$$

and substituting this in the equation for ε, we get

$$\varepsilon = \frac{1}{2}\mu \underline{\dot{r}}^2 + \underbrace{\frac{1}{2}\frac{\mu h^2}{r^2} - \frac{Gm_1m_2}{r}}_{\text{effective potential}}$$

$\underline{r} = r\underline{e_r} \Rightarrow \frac{GMr\underline{e_r}}{r^3}$

$\Rightarrow \frac{GM}{r^2}\underline{e_r}$

Solving the Equation of Motion Polar

The equation of motion for this is

$$\begin{aligned}
\underline{\ddot{r}} = -\frac{GM}{r^2}\underline{e_r} &\Rightarrow (\ddot{r} - r\dot{\theta}^2)\underline{e_r} = -\frac{GM}{r^2}\underline{e_r} \\
&\Rightarrow (\ddot{r} - r\dot{\theta}^2) = -\frac{GM}{r^2} \\
&\Rightarrow \ddot{r} - \frac{h^2}{r^3} = -\frac{GM}{r^2}
\end{aligned}$$

Note (Nice Trick for solving the differential equation). Put

$$r = \frac{1}{u} \qquad u = u(\theta) \text{ and } \theta = \theta(t)$$

and therefore finding the first and second derivatives:

1.
$$\begin{aligned}
\dot{r} &= \frac{dr}{dt} \\
&= \frac{d}{dt}\left(\frac{1}{u}\right) \\
&= \frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{1}{u}\right) \\
&= \dot{\theta}\left(-\frac{1}{u^2}\right)\frac{du}{d\theta} \\
&= hu^2\left(-\frac{1}{u^2}\right)\frac{du}{d\theta} \\
&= -h\frac{du}{d\theta}
\end{aligned}$$

2.
$$\begin{aligned}
\ddot{r} &= \frac{d^2r}{dt^2} \\
&= \dot{\theta}\frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right) \\
&= hu^2\left(-h\frac{d^2u}{d\theta^2}\right) \\
&= -h^2u^2\frac{d^2u}{d\theta^2}
\end{aligned}$$

Therefore the equation of motion becomes

$$-hu^2\frac{d^2u}{d\theta^2} - h^2u^3 = -GMu^2 \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

. Therefore we **need to solve** this **homogeneous second order differential equation**

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} \tag{2.34}$$

Solving (2.34) using Ansatz $u = e^{\lambda\theta}$ first year calculus

$$u = A\cos(\theta - \theta_0) + \frac{GM}{h^2}$$

Or more conviently, we can write

$$\boxed{r = \frac{1}{u} \Rightarrow u = \frac{1}{r} = \frac{GM}{h^2}\left(1 + e\cos(\theta - \theta_0)\right)} \qquad (*r)$$

where e is the **eccentricity** of the orbit and θ_0 is the **true anomaly**.

Checking if Energy is conserved

$$\mathcal{E} = \frac{1}{2}\mu h^2 \left(\frac{GM}{h^2}\right)^2 e^2 \sin^2(\theta - \theta_0)$$
$$+ \frac{1}{2}\mu h^2 \left(\frac{GM}{h^2}\right)^2 \left(1 + 2e\cos(\theta-\theta_0) + e^2\cos^2(\theta-\theta_0)\right)$$
$$- G\mu M \frac{GM}{h^2}\left(1 + e\cos(\theta - \theta_0)\right)$$

$$\Rightarrow \mathcal{E} = \frac{1}{2}\mu \frac{(GM)^2}{h^2}\left(e^2 - 1\right) = \text{CONSTANT} \qquad (*\mathcal{E})$$

And therefore we can say energy is constant

(continued on next page)

Given $(*r)$,

$$\boxed{r = \frac{l}{1 + e\cos\theta}} \qquad \boxed{r^2\dot{\theta} = h}$$

$$\boxed{l = \frac{h^2}{GM} \Rightarrow \dot{\theta} = \frac{\sqrt{GMl}}{r^2}} \Rightarrow \boxed{r^2\dot{\theta} = \sqrt{GMl}}$$

Differentiating r, (quotient rule, $\theta = \theta(t)$)

$$\dot{r} = \frac{le\dot{\theta}\sin\theta}{(1+e\cos\theta)^2} = \frac{le\sqrt{GMl}}{r^2}\,\frac{\sin\theta}{(1+e\cos\theta)^2} = \sqrt{\frac{GM}{l}}\,e\sin\theta$$

$$\Rightarrow \boxed{\dot{r} = \sqrt{\frac{GM}{l}}\,e\sin\theta}$$

Finding speed; As seen in polar section,

$$|\dot{\underline{r}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \frac{GM}{l}\left(e^2\sin^2\theta + (1+e\cos\theta)^2\right) = \frac{GM}{l}(1+2e\cos\theta + e^2)$$

$$\Rightarrow \boxed{|\dot{\underline{r}}|^2 = \frac{GM}{l}(1+2e\cos\theta + e^2)}$$

max speed when $\cos\theta = 1$

min speed when $\cos\theta = -1$

Finding energy of orbit

$$\mathcal{E} = \frac{1}{2}\mu|\dot{\underline{r}}|^2 - \frac{\mu M G}{r} = \frac{\mu G M}{2l}(1+2e\cos\theta + e^2) - \frac{\mu M G}{l}(1+e\cos\theta)$$

$$= \frac{\mu G M}{2l}(e^2-1)$$

$$\Rightarrow \boxed{\mathcal{E} = \frac{\mu G M}{2l}(e^2-1)}$$

Normal to take the eccentricity $e>0$:

- if $e^2<1$ then $\varepsilon<0$ $(0<e<1)$
- if $e^2=1$ then $\varepsilon=0$ $(e=1)$
- if $e^2>1$ then $\varepsilon>0$ $(e>1)$

Properties of Orbit

(i) $e=0$:

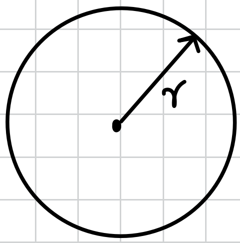
if $e=0$ then $\boxed{u=\frac{1}{r}=\frac{GM}{h^2}}$ or $\boxed{r=\frac{h^2}{GM}}$ (r is constant $\Rightarrow$ circular)

Here since r is constant, we have a circular orbit.

Further put

$$(*1)\ \boxed{l=\frac{h^2}{GM}} \Longrightarrow r=l \Rightarrow \boxed{l=\text{radius of orbit}}$$

Furthermore, calculating $\dot{\theta}$



Circumference $2\pi r$

Time to go all the way round is T, then

$$\dot{\theta}=\frac{2\pi}{T}$$

$$\dot{\theta}=\frac{h}{l^2}=\frac{2\pi}{T}$$

$$\Longrightarrow \boxed{T=\frac{2\pi}{\sqrt{GM}}\,l^{3/2}}$$

where T is the orbital time.

(ii) $0<e<1$

Here $\boxed{r=\frac{l}{1+e\cos\theta}}$ $(*2)$ $\left(\text{from } (*1)\ l=\frac{h^2}{GM} \text{ and } u=1+e\cos(\theta-\theta_0)\right)$

$$r^2\dot{\theta}=\sqrt{GMl}$$

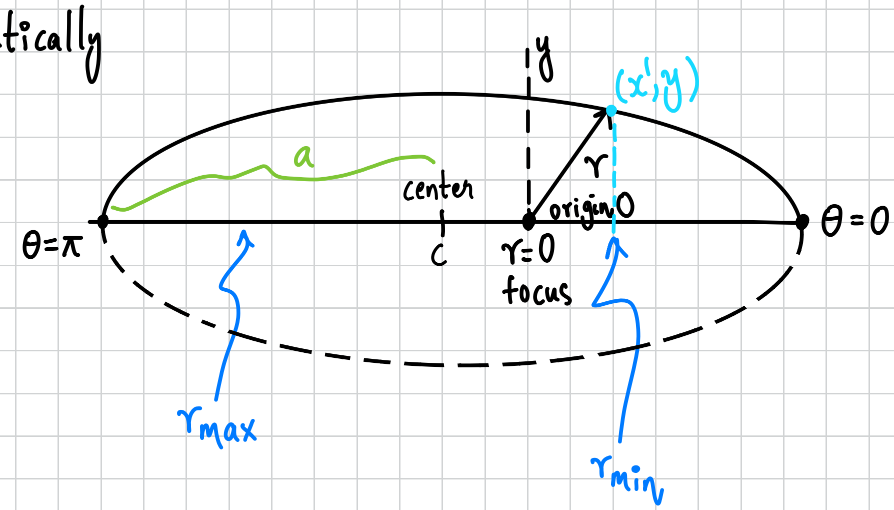
Since in $(*2)$, $0<e<1$, the denominator can never disappear. Further, r is periodic and bounded.

(a) Orbit is periodic and bounded

$$\theta=\pi \Longrightarrow r_{max}=\frac{l}{1-e}$$ periodic so same at $\theta=\pi, 3\pi, 5\pi, \ldots$

$$\theta=0 \Longrightarrow r_{min}=\frac{l}{1+e}$$ periodic so same at $\theta=0, 2\pi, 4\pi, 6\pi, \ldots$

Diagramatically



Total width of the orbit

$$r_{min} + r_{max} = \frac{l}{1-e} + \frac{l}{1+e}$$

$$= \frac{2l}{1-e^2}$$

Define

(*a1) $\boxed{a = \frac{l}{1-e^2}}$ $\Longrightarrow$ width $= 2a$ $\Longrightarrow$ $\boxed{a = \frac{r_{min} + r_{max}}{2}}$

The distance from center to origin: C to O

$$a - r_{min} = \frac{l}{1-e^2} - \frac{l}{1+e}$$

$$= \frac{el}{1-e^2}$$

$$= ea$$

$\Rightarrow$ $\boxed{\text{distance from origin to center} = ea}$

Converting to Cartesian Co Ordinates

Choosing origin situated at $r=0$

Note: From (*2) $r + \underbrace{er\cos\theta}_{x'} = l$

as from polar section $x' = r\cos\theta$

or $\boxed{r = l - ex'}$ (*3)

Squaring (*3)

$$r^2 = \overbrace{x'^2 + y^2}^{\text{from diagram}} = \underbrace{(l - ex')^2}_{\text{quadratic equation}}$$

$$\Rightarrow \quad r^2 = x'^2 + y^2 = \ell^2 - 2\ell e x' + e^2 x'^2$$

Completing the square

$$y^2 + (1-e^2)x'^2 + 2\ell e x' = \ell^2$$

$$\Rightarrow \quad y^2 + (1-e^2)\Big(x' + \underbrace{\frac{\ell e}{1-e^2}}_{ea}\Big) - \frac{\ell^2 e^2}{1-e^2} = \ell^2$$

$$\Rightarrow \quad y^2 + (1-e^2)(x'+ea)^2 = \ell^2\left[\cancel{e^2} + 1 - \cancel{e^2}\right]\frac{1}{1-e^2}$$

And multiplying the final equation through by $\frac{1-e^2}{\ell^2}$, we get

$$\boxed{\frac{y^2}{b^2} + \frac{(x'+ea)^2}{a^2} = 1} \qquad \text{where } b^2 = \frac{\ell^2}{1-e^2} \Longrightarrow \boxed{b = a\sqrt{1-e^2}}$$

(*b1)

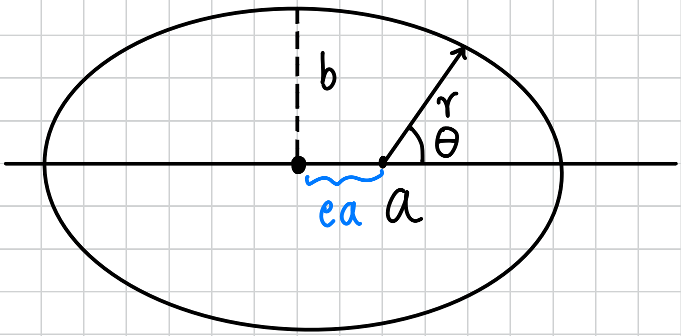
equation of an ellipse.

Redefine $x = x' + ea$ to get

$$\boxed{\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1}$$

So shape of orbit is an ellipse

one of keplers laws



Orbital Period (use $r^2\dot{\theta} = \sqrt{GM\ell}$)

Let T be the orbital period (period of one orbit)

Integrating
$$\int_0^T \sqrt{GM\ell}\, dt = T\sqrt{GM\ell} = \int_0^T r^2\dot{\theta}\, dt = \boxed{\int_0^{2\pi} r^2\, d\theta}$$

formula for area using polar co-ordinates

change of variables

Therefore

area of ellipse

$$T\sqrt{GM\ell} = 2 \times \text{area} \Rightarrow T\sqrt{GM\ell} = 2\pi ab$$

$$\Rightarrow \boxed{T = \frac{2\pi ab}{\sqrt{GM\ell}} = \frac{2\pi}{\sqrt{GM\ell}} \frac{\ell}{1-e^2} \frac{\ell}{\sqrt{1-e^2}}}$$

$$\Rightarrow T = \frac{2\pi a^{3/2} \cancel{\sqrt{\ell}}}{\sqrt{GM\cancel{\ell}}}$$

$$\Rightarrow \boxed{T = \frac{2\pi a^{3/2}}{\sqrt{GM}}}$$

Definition: Orbital Period for elliptical orbit

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

where a is the distance from center of elliptical orbit from center to the furthest or closest point.

Using this, we get one of Keplers laws

$$T^2 = \frac{4\pi^2}{GM} a^3 \Rightarrow \boxed{T^2 \propto a^3}$$

Note: For planet-sun, $M \sim M_{sun}$ so

$$\frac{4\pi^2}{GM}$$

same for every planet

Some numbers

1) $G = 6.67 \times 10^{-11} \ Nm^2/kg^2$

2) $M_{sun} = 1.989 \times 10^{30}$ kg

3) $M_{earth} = 5.972 \times 10^{24}$ kg

4) $M_{moon} = 7.348 \times 10^{22}$ kg

Orbital properties for earth-sun

1) $e_{Earth} = 0.017$

2) $r_{max} = 152.1 \times 10^9 m$

3) $r_{min} = 147.1 \times 10^9 m$

4) $\frac{M_{Earth}}{M_{Sun}} \simeq 3 \times 10^{-6}$

5) $\mu_E = \frac{m_S m_E}{m_S + m_E} \simeq m_E$

since $\begin{rcases} \underline{r}_1 = \underline{R} + \underline{s}_1 \\ \underline{r}_2 = \underline{R} + \underline{s}_2 \end{rcases} \Rightarrow \underline{s}_{sun} = -\frac{m_E \underline{r}}{m_E + m_S}, \quad \underline{s}_E = \frac{m_S \underline{r}}{m_E + m_S}$ (from notes above)

6) $|\underline{s}_{sun}| \simeq \frac{m_E}{m_S} |\underline{r}| \simeq 450 \times 10^3 m = 450 km$

Energy in Elliptical Orbit

From $(*\varepsilon)$

$$\varepsilon = \frac{1}{2}\mu \frac{(GM)^2}{h^2}(e^2-1)$$

$$= \frac{1}{2}\mu \frac{(GM)^2}{GM\ell}(e^2-1)$$

$$= -\frac{1}{2}\mu \frac{GM}{a}$$

$$= -\frac{1}{2} G \frac{m_1 m_2}{a} \quad Nm$$

Note from before:

$$r^2\dot{\theta} = h = \sqrt{GM\ell}$$

$(*r1)$

$\left(\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad a = \frac{\ell}{1-e^2}\right)$

$$\Rightarrow \boxed{\varepsilon = -\frac{1}{2} G_N \frac{m_1 m_2}{a}} \qquad (*\varepsilon 2)$$

Calculating ε in sun-earth system

$$\varepsilon = -\frac{1}{2} \times \frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30} \times 5.97 \times 10^{24}}{1.5 \times 10^{11}} \simeq -2.6 \times 10^{33} Nm$$

(iii) $e>1$

Here $\boxed{\dfrac{l}{r} = 1 + e\cos\theta}$ (*4) $\left(\text{from } (*1)\ l = \dfrac{h^2}{GM},\ u = 1 + e\cos(\theta - \theta_0) \right)$

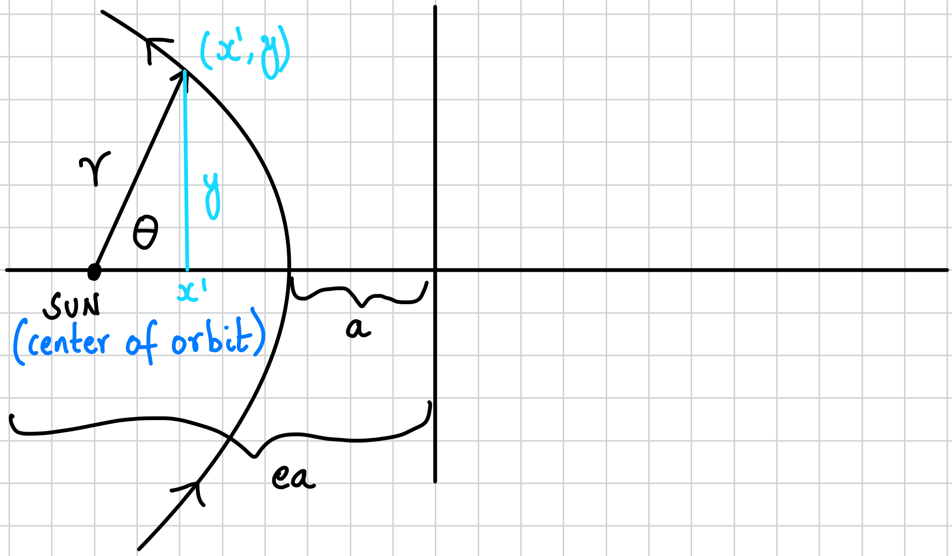
Note: The fact that

$$1 + e\cos\theta = 0 \text{ when } \cos\theta = -\frac{1}{e} \text{ and } r \to \infty$$

So in effect what happens is that the particle starts at infinity, comes towards the sun and then goes off again back to infinity.

So orbit starts and ends at ∞

Diagram of orbit below



Here $\boxed{r_{min} = \dfrac{l}{1+e}}$ $\boxed{a = \dfrac{l}{e^2 - 1}}$ (*a2)

Converting to Cartesian Co Ordinates

Note: From (*2) $\underbrace{r + er\cos\theta}_{x'} = l$ as from polar section $x' = r\cos\theta$

or $\boxed{r = l - ex'}$ (*3)

Squaring (*3)

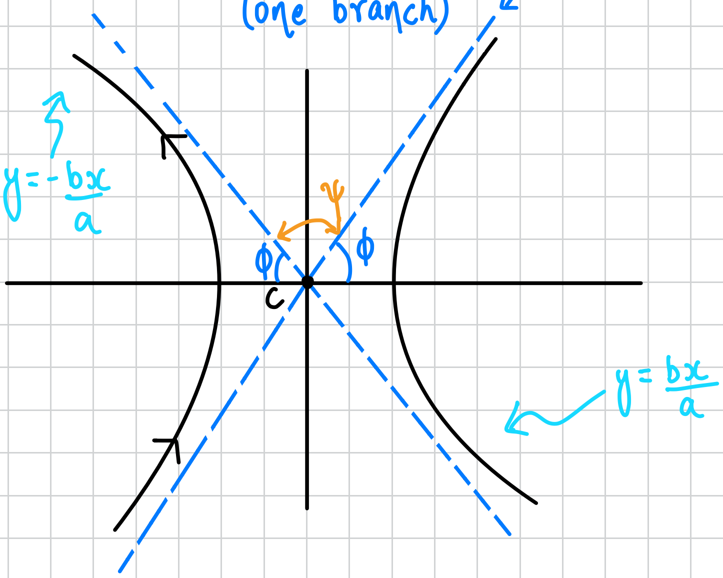
$$r^2 = x'^2 + y^2 = (l - ex')^2$$

$$\Rightarrow r^2 = x'^2 + y^2 = l^2 + e^2x'^2 - 2lex'$$

Reorganize similar for case (ii)

$$\boxed{\frac{(x'-ea)^2}{a^2} - \frac{y^2}{b^2} = 1} \quad \text{where} \quad \boxed{b = a\sqrt{e^2-1}} \quad (*b2)$$

equation of hyperbola (one branch)



putting $x = x' - ea$

$$\boxed{y = \pm \frac{bx}{a}}$$

(asymptotes)

Calculating angle of Orbit (angle ψ above)

Computing angle ψ, from the diagram above

$$2\phi + \psi = \pi \Rightarrow \phi = \frac{\pi}{2} - \frac{\psi}{2}$$

$$\Rightarrow \psi = \pi - 2\phi$$

But from the fact that $\tan\phi = b/a$

$$\boxed{\psi = \pi - 2\tan^{-1}\frac{b}{a}}$$

Also $b/a = \sqrt{e^2-1}$, so

$$\tan^2\phi = e^2 - 1 \Rightarrow \cos\phi = \frac{1}{e}$$

Therefore

$$\boxed{\psi = \pi - 2\cos^{-1}\left(\frac{1}{e}\right)}$$

Definition: Scattering angle

$$\psi = \pi - 2\cos^{-1}\left(\frac{1}{e}\right)$$

(iv) $e = 1$

Here $\boxed{\frac{l}{r} = 1 + \cos\theta}$ (*5) $= 2\cos^2\left(\frac{\theta}{2}\right)$

Note: The fact that

$1 + \cos\theta = 0$ when $\theta = \pm\pi \implies r \to \infty$.

Similar to above

So orbit starts and ends at ∞

Same idea as with other cases

$r = l - x'$ (and using that $x' = r\cos\theta$)

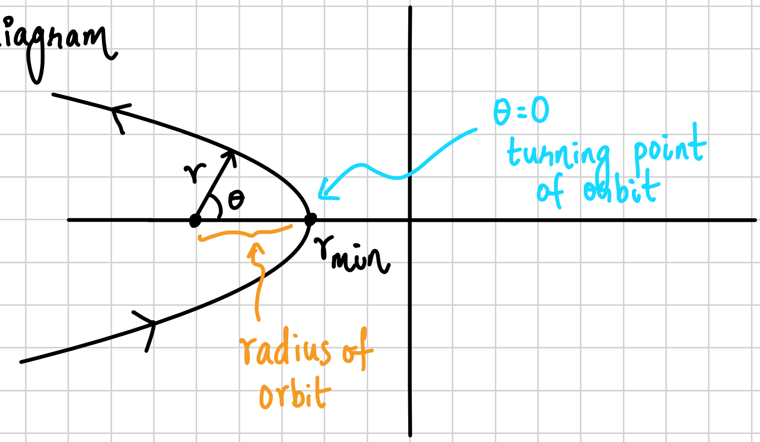
We have that

$$\cancel{x'^2} + y^2 = l^2 - 2lx' + \cancel{x'^2}$$

$$\implies \boxed{y^2 = l(l - 2x')}$$

equation of parabola (one branch)

Drawing diagram



No center as $e = 1$

$$\boxed{r_{min} = \frac{l}{2}}$$ (when $\theta = 0$)

Calculating Speed of Orbit

We know that (2.6)

$$|\dot{r}|^2 = u^2 = \dot{r}^2 + r^2\dot{\theta}^2, \quad \text{so}$$

(Note that from (*5) and (*r1) $r = \frac{l}{1+\cos\theta}$ $\quad r^2\dot{\theta} = \sqrt{GMl}$)

So

$$\dot{r} = \frac{l\dot{\theta}\sin\theta}{(1+\cos\theta)^2} = \frac{r^2}{l}\dot{\theta}\sin\theta = \frac{h}{l}\sin\theta$$

Here $\dot{r}$ is minimum when $\theta=0$ $\Rightarrow$ turning point of orbit, then we get

$$\boxed{\dot{r}^2+r^2\dot{\theta}^2=\frac{2GM}{r_{min}}\cos^2\frac{\theta}{2}}$$

(max at $\theta=0$)

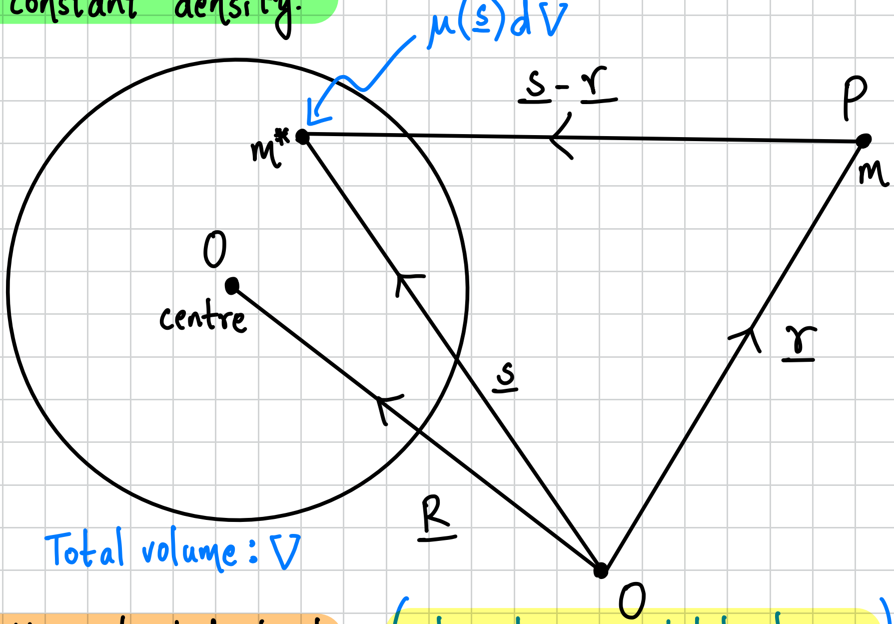
square of the escape velocity

Definition: Escape Velocity of parabolic orbit

$$|\dot{\underline{r}}|^2=\dot{r}^2+r^2\dot{\theta}^2=\frac{2GM}{r_{min}}\cos^2\frac{\theta}{2}$$

2.12 Gravitational Potential Revisited

i Consider the sun as a sphere of constant density.



Here

- $\mu(\underline{s})$ is the mass density
- dV is the volume element.

Therefore mass at that point is:

mass = density × volume

$$\Rightarrow \boxed{m^*=\mu(\underline{s})\,dV}$$

The gravitational potential due to the extended object is (integrating over total volume V)

$$\boxed{\Phi=\int_V \frac{mG\mu(\underline{s})\,dV}{|\underline{r}-\underline{s}|}} \quad (*p1)$$

The total mass of the object of volume V is

$$\boxed{M=\int_V \mu(\underline{s})\,dV} \quad \text{and} \quad \boxed{M\underline{R}=\int_V \mu(\underline{s})\,\underline{s}\,dV}$$

Consider following conditions

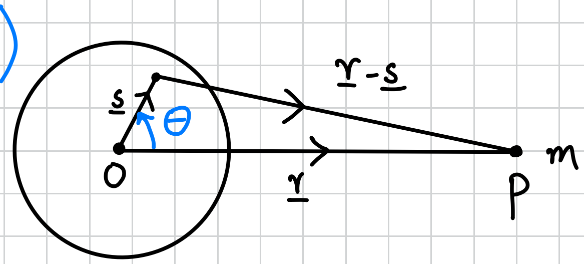
(i) Constant mass density sphere of radius a

(ii) Put the origin at center of mass

at center of sphere as mass density is constant

$\mu(\underline{s})$ constant, say $\boxed{\mu(\underline{s})=\mu_0}$

(use polar co-ordinates with $\underline{r}$ as the axis to measure θ)



Then

$$\boxed{\Phi = G\mu_0 m \int_{\text{sphere}} \frac{dV}{|\underline{r}-\underline{s}|}} \quad (*p2)$$

Choose spherical polars so $\underline{r}$ is the polar axis (i.e. $\theta = 0$ is the direction of $\underline{r}$)

Then

$$|\underline{r}-\underline{s}|^2 = (\underline{r}-\underline{s})\cdot(\underline{r}-\underline{s}) = |\underline{r}|^2 - 2\underline{r}\cdot\underline{s} + |\underline{s}|^2$$

scalar product

$$= |\underline{r}|^2 - 2|\underline{r}||\underline{s}|\cos\theta + |\underline{s}|^2$$

$$\equiv r^2 - 2rs\cos\theta + s^2 \qquad (|\underline{r}| = r, \quad |\underline{s}| = s)$$

Therefore we get

$$\Phi = G\mu_0 m \int_{\text{sphere}} \frac{s^2 \sin\theta \, ds\, d\theta\, d\phi}{(r^2 - 2rs\cos\theta + s^2)^{1/2}}$$

volume element $dV = s^2 \sin\theta \, ds\, d\theta\, d\phi$

Boundary conditions

$0 \le s \le a$; $0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

And we get the following conditions

$$\Phi = G\mu_0 m \int_{\text{sphere}} \frac{s^2 \sin\theta \, ds\, d\theta\, d\phi}{(r^2 - 2rs\cos\theta + s^2)^{1/2}} = \mu_0 m G 2\pi \int_0^a \frac{s^2\, ds}{rs} \left[(r^2 - 2rs\cos\theta + s^2) \right]_0^\pi$$

$$= \mu_0 m G 2\pi \frac{1}{r} \int_0^a s \, ds \left[((r+s)^2)^{1/2} - ((r-s)^2)^{1/2} \right] \quad (*e1)$$

could be negative, do not take square roots directly. Consider 2 possibilities

Two possibilities

a) $r > a$:

$r > a \Rightarrow r > a > s$

$\Rightarrow$ take positive square roots $(*e1)$ becomes

$$\Rightarrow \quad \Phi = \mu_0 G m \frac{2\pi}{r} \int_0^a 2s^2 \, ds$$

$$= G m \mu_0 \frac{2\pi}{r} \frac{2a^3}{3}$$

$$= G m \mu_0 \frac{4\pi a^3}{3} \frac{1}{r}$$

volume of sphere

$\mu_0 \times$ volume $=$ total mass

density $\times$ volume $=$ mass

$$\Rightarrow \boxed{\Phi = \frac{GmM}{r}} \quad (*p2)$$

b) $r<a$: Split integral into 2 parts.

$$\Phi = G\mu_0 m \frac{2\pi}{r}\left(\underbrace{\int_r^a 2rs\, ds}_{s>r} + \underbrace{\int_0^r 2s^2 ds}_{s<r}\right)$$

$$= G\mu_0 m \frac{2\pi}{r}\left(2r\frac{1}{2}(a^2-r^2) + \frac{2}{3}r^3\right)$$

$$= G\mu_0 m 2\pi\left(a^2 - r^2 + \frac{2}{3}r^2\right)$$

$$= G\mu_0 m 2\pi\left(a^2 - \frac{r^2}{3}\right)$$

$$= \frac{3GMm}{2a}\left(1 - \frac{r^2}{3a^2}\right)$$

$$\Longrightarrow \boxed{\Phi = \frac{3GMm}{2a}\left(1-\frac{r^2}{3a^2}\right)} \quad (*p3)$$

This is when $r<a$, that is when object is inside the sphere

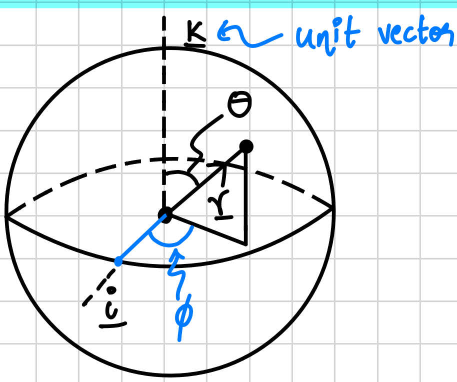
Gradient of potential

$$\nabla\Phi = \frac{-GmM}{a^3} \quad \text{at } r=a$$

Spherical Polars

The components of $\mathbf{r}$ are

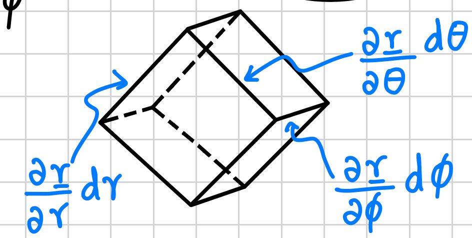
$$\boxed{\mathbf{r} = r(\underbrace{\sin\theta\cos\phi\,\mathbf{i}}_{x} + \underbrace{\sin\theta\sin\phi\,\mathbf{j}}_{y} + \underbrace{\cos\theta\,\mathbf{k}}_{z})}$$



Calculating volume element: dV

Think of infinitesimal changes of r, θ and ϕ

$$\frac{\partial \mathbf{r}}{\partial r}dr, \quad \frac{\partial \mathbf{r}}{\partial \theta}d\theta, \quad \frac{\partial \mathbf{r}}{\partial \phi}d\phi$$



If we want to work out volume occupied by above

$$\boxed{\frac{\partial \underline{r}}{\partial r} \cdot \left(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \right) dr\, d\theta\, d\theta}$$ ← triple scalar product (*s1)

Equation (*s1) can be calculated by

$$\frac{\partial \underline{r}}{\partial r} \cdot \left(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \right) dr\, d\theta\, d\theta \equiv \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} dr\, d\theta\, d\phi$$

← Jacobian matrix

$$= \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

expand determinant from bottom row

$$= -r\sin\theta\sin\phi\,(-r\sin\theta) - r\sin\theta\cos\phi\,(-r\cos\phi)$$

$$= r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\Rightarrow \boxed{\frac{\partial \underline{r}}{\partial r} \cdot \left(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \right) dr\, d\theta\, d\theta = r^2\sin\theta\, dr\, d\theta\, d\phi}$$

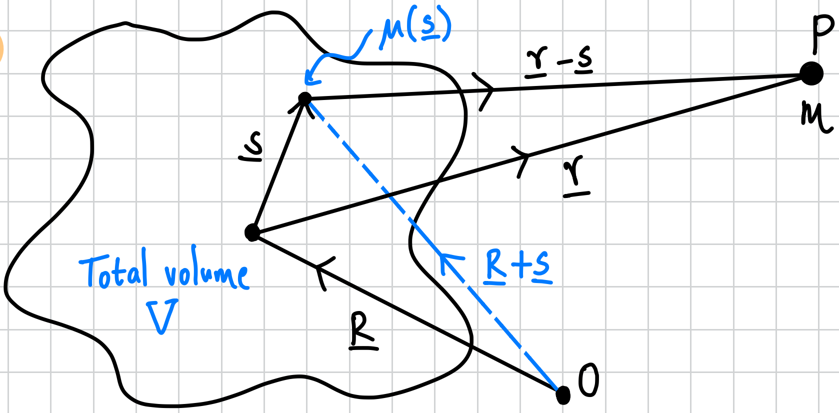
Boundary conditions: 1) $r \geq 0$

2) $0 \leq \theta \leq \pi$

3) $0 \leq \phi \leq 2\pi$

2.12 Gravitational Potential Revisited (contd.)

(ii) Consider an object that may not have constant density



Let
- $\mu(\underline{s})$ be mass density
- dV be volume element
- Total volume is V

Note: $\mu(\underline{s})$ is the mass density at $\underline{s}$ relative to $\underline{R}$ the center of mass.

Therefore the gravitational potential

$$\boxed{\Phi = Gm \int_V \frac{\mu(\underline{s})}{|\underline{r}-\underline{s}|}\, dV}$$ (*p4)

Consider $|\underline{r}|$ to be much greater than the scale of V ($|\underline{r}| \gg |\underline{s}|$). Therefore

$$\frac{1}{|\underline{r}-\underline{s}|} = \frac{1}{|\underline{r}|} - \underline{s}\cdot\underline{\nabla}\left(\frac{1}{|\underline{r}|}\right) + \frac{1}{2}(\underline{s}\cdot\underline{\nabla})(\underline{s}\cdot\underline{\nabla})\left(\frac{1}{|\underline{r}|}\right) + \cdots$$ (3 dimensional taylor series)

Note: Taylor Series

$$\boxed{f(x+h) = f(x) + hf'(x) + \frac{1}{2!}h^2 f''(x) + \cdots}$$

Here for 3D:

- $x = \underline{r}$
- $h = -\underline{s}$ (increment)
- $h \times$ derivative is increment $\cdot$ gradient

scalar product

Therefore inserting into integral (*p4)

$$\Phi(\underline{r}) = Gm\left[\int_V \frac{\mu(\underline{s})}{|\underline{r}|}dV - \frac{\partial}{\partial x_a}\left(\frac{1}{|\underline{r}|}\right)\int_V s_a \mu(\underline{s})\, dV + \frac{1}{2}\frac{\partial}{\partial x_a}\frac{\partial}{\partial x_b}\frac{1}{|\underline{r}|}\int_V s_a s_b \mu(\underline{s})\, dV + \cdots\right]$$

(used einstein notation for $\underline{\nabla}$ gradient formula)

from above $\int_V \mu(\underline{s})\, dV = \text{total mass} = M$

$$= \boxed{\frac{GmM}{|\underline{r}|}} + 0 + \left(\frac{1}{|\underline{r}|^5}\right)\left(-|\underline{r}|^2\delta_{ab} + 3x_a x_b\right)\int s_a s_b \mu(\underline{s})\, dV$$

as $M\underline{R} = \int_V \mu(\underline{s})(\underline{R}+\underline{s})\, dV$

$$\Rightarrow \int_V \underline{s}\mu(\underline{s})\, dV \equiv 0$$

Similar to discrete case proof

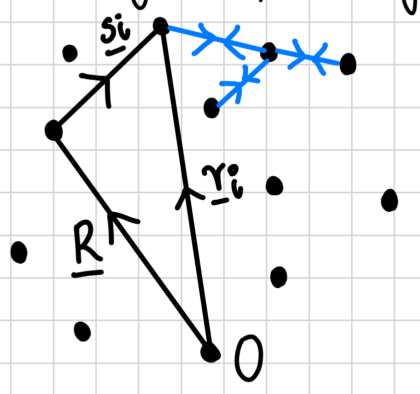
$$\sum m_i \underline{s}_i = 0$$

Justifies our assumptions when $|\underline{r}| \gg 0$ (very big)

2.13 Rigid Bodies

This is an idealization of inter-particle forces such as to maintain rigid shape of body.

Consider a collection of particles. We have



$|\underline{r_i} - \underline{r_j}|$ is constant. Since body is rigid, the locations of particles do not change

$\Rightarrow |\underline{s_i} - \underline{s_j}|$ is constant

$|\underline{s_i}|$ is constant as $\underline{r_i} = \underline{R} + \underline{s_i}$

Further, assume the inter-particle forces have the property

$$\boxed{\underline{F_{ij}} \propto \underline{r_j} - \underline{r_i} = -\underline{F_{ji}}}$$

Force is in direction joing 2 particles.

Total Kinetic Energy and Angular Momentum

We can calculate the total Kinetic energy

using that $\underline{r_i} = \underline{R} + \underline{s_i}$

$$\boxed{K_{tot} = \frac{1}{2}\sum_{i=1}^{N}\frac{1}{2}m_i|\underline{\dot{r}_i}|^2 = \frac{1}{2}M|\underline{\dot{R}}|^2 + \frac{1}{2}\sum_{i=1}^{N} m_i|\underline{\dot{s}_i}|^2}$$

Simililarly calculating Total Angular Momentum

use that $\underline{r_i} = \underline{R} + \underline{s_i}$

$$\boxed{\underline{J}_{tot} = \sum_{i=1}^{N} m_i \underline{r_i} \times \underline{\dot{r}_i} = M\underline{R} \times \underline{\dot{R}} + \sum_{i=1}^{N} m_i \underline{s_i} \times \underline{\dot{s}_i}}$$

Now $|\underline{s_i}|$ is fixed but the rigid body can rotate so can have a velocity associated with it.
Therefore $\underline{s_i}$ is generally time dependant.

Let

$$\underline{s_i}(t) = \underbrace{s_{ia}(t)}_{\text{time dependant}}\,\underline{e}_a^{(0)} \quad \text{fixed in space} \quad \text{(fixed axes)} \quad \text{(Einstein's Notation)}$$

$$= \underbrace{s_{ia}^{(0)}}_{\text{constants}}\,\underline{e}_a(t) \quad \text{fixed relative to rigid body} \quad \text{(axes moving with body)}$$

Change of axes:
Changing the to be from the body in motion, i.e. the origin is on the body instead of at a distance.

- When origin on body the s_{ia} will be constant but the unit vectors rotate/change hence time dependant
- When origin away, unit vectors constant, s_{ia} is moving

Therefore the two axes

- one on moving body (*a1)
- other at a fixed at a distance (*a2)

The axes (*a1) and (*a2) are translated and rotated from each other.

We consider the rotation, we can say

(*a3) $\boxed{\underline{e}_a(t) = \underbrace{R_{ab}(t)}_{\text{matrix}}\, \underline{e}_b^{(0)}}$; $\boxed{RR^T = \mathbb{1}, \quad \det(R) = 1}$

Then differentiating with respect to time

re expressed fixed ones using unfixed ones by inverting (*a3)
↳ taking transpose

$$\dot{\underline{e}}_a(t) = \dot{R}_{ab}(t)\,\underline{e}_b^{(0)} = \dot{R}_{ab}(t)\, R^T_{bc}(t)\,\underline{e}_c(t)$$

Note:

$$RR^T = \mathbb{1} \Rightarrow \frac{d}{dt}(RR^T) = 0 \Rightarrow \dot{R}R^T + R\dot{R}^T = 0$$

$$\Rightarrow \boxed{\dot{R}R^T = -(\dot{R}R^T)^T} \qquad \left[(AB)^T = B^T A^T\right]$$

This means that $\dot{R}R^T$ is antisymmetric hence

$$\boxed{(\dot{R}R^T)_{ab} = \varepsilon_{abc}\,\omega_c(t)} \quad (*a4)$$

Example calculating rotation matrix

Let

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ and therefore we get

$$\dot{R}R^T = \begin{pmatrix} -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta & 0 \\ -\dot{\theta}\cos\theta & -\dot{\theta}\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \overset{\omega_3}{\dot{\theta}} & 0 \\ -\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore so far, we have found that substituting (*a4) into (*a3)

$$\boxed{\dot{\underline{e}}_a(t) = \varepsilon_{acd}\, \omega_d(t)\, \underline{e}_c(t)} \quad (*a5)$$

Hence

$$\dot{\underline{s}}_i(t) = s_{ia}^{(0)}\, \dot{\underline{e}}_a(t)$$

$$= s_{ia}^{(0)}\, \varepsilon_{acd}\, \omega_d(t)\, \underline{e}_c(t)$$

reorder to ε_{dac} and $f \to c$

$$\left(\underline{\omega} \times \underline{s}_i = \omega_d \underline{e}_d(t) \times \underline{s}_{ia}^{(0)}\, \underline{e}_a(t) = \omega_d(t)\, \varepsilon_{daf}\, \underline{e}_f(t)\, s_{ia}^{(0)} \right)$$

$$= \underline{\omega} \times \underline{s}_i$$

indicates ω is angular velocity

$$\Rightarrow \boxed{\dot{\underline{s}}_i(t) = \underline{\omega} \times \underline{s}_i}$$

Calculating Total Kinetic Energy

As seen above,

$$K_{tot} = \frac{1}{2} M |\dot{\underline{R}}|^2 + \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{\underline{s}}_i|^2$$

$$= \frac{1}{2} M |\dot{\underline{R}}|^2 + \frac{1}{2} \sum_{i=1}^{N} m_i |\underline{\omega} \times \underline{s}_i|^2$$

$$(\underline{\omega} \times \underline{s}_i) \cdot (\underline{\omega} \times \underline{s}_i) = \underline{\omega} \cdot (\underline{s}_i \times (\underline{\omega} \times \underline{s}_i))$$

$$= \underline{\omega} \cdot ((\underline{s}_i \cdot \underline{s}_i)\underline{\omega} - (\underline{s}_i \cdot \underline{\omega})\underline{s}_i)$$

$$= \omega_a(t) \left(|\underline{s}_i|^2 \omega_a - s_{ib} \omega_b s_{ia} \right) \quad \text{(Einstein Notation)}$$

$$= \boxed{\left(|s_i|^2 \delta_{ab} - s_{ia} s_{ib} \right) \omega_a \omega_b}$$

Therefore total kinetic energy is (moment of inertia tensor)

$$\boxed{K_{tot} = \frac{1}{2} M |\dot{\underline{R}}|^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \left(|\underline{s}_i|^2 \delta_{ab} - s_{ia}^{(0)} s_{ib}^{(0)} \right) \omega_a \omega_b} \quad (*t_1)$$

This is called the moment of inertia tensor.

Therefore in the end we define the inertia tensor

$$\boxed{I_{ab} = \sum_i m_i \left(|\underline{s_i}|^2 \delta_{ab} - s_{ia}^{(0)} s_{ib}^{(0)} \right)}$$

(*t2) This is a constant matrix

Note: The inertia tensor needs to be defined for each rigid body

Note: The inertia tensor is symmetrical: $\boxed{I_{ab} = I_{ba}}$ The eigenvalues (principal moment of inertia) are real ($\mathbb{R}$)

Note: We can choose $\underline{e}a(t)$ to be orthormal eigenvectors I_{ab}. Hence I_{ab} is a diagonal matrix (called principal axes) (or principal moments of inertia)

Calculating Total Angular Momentum

Angular Momentum as seen above

$$\underline{J}_{tot} = M\underline{R} \times \dot{\underline{R}} + \sum_{i=1}^{N} m_i \underline{s_i} \times \dot{\underline{s_i}}$$

To make simplification easier,

$$\underline{J}_{tot} = M\underline{R} \times \dot{\underline{R}} + \underline{L}, \qquad \boxed{\underline{L} = \sum_i m_i \underline{s_i} \times \dot{\underline{s_i}}}$$

As seen from the calculations above,

$$\dot{\underline{s_i}} = \underline{\omega} \times \underline{s_i} \implies \underline{L} = \sum_i m_i \underline{s_i} \times (\underline{\omega} \times \underline{s_i})$$

$$\implies \underline{L} = \sum_i m_i \left(|\underline{s_i}|^2 \underline{\omega} - (\underline{s} \cdot \underline{\omega}) \underline{s_i} \right) \quad (\text{*e2})$$

So calculating the a^{th} component of $\underline{L}$ in (*e2): L_a

Kronecker δ properties
$\delta_{ac} w_c = w_a$
and change dummy index
$s_{ib} w_b = s_{ic} w_c$

a^{th} component of vectors

$$\underline{L_a} = \sum_{i=1}^{N} m_i \Big(\underbrace{s_{ib}^{(0)} s_{ib}^{(0)}}_{|\underline{s_i}|^2} \underline{w_a} - \underbrace{s_{ib}^{(0)} w_b}_{\underline{s} \cdot \underline{\omega}} \underline{s_{ia}^{(0)}} \Big) = \sum_{i=1}^{N} \left(s_{ib}^{(0)} s_{ib}^{(0)} \delta_{ac} - s_{ia}^{(0)} s_{ic}^{(0)} \right) w_c$$

$$= I_{ac} w_c$$

$$\implies \boxed{L_a = I_{ac} w_c} \quad (\text{*t}_3) \quad \implies \boxed{\underline{L} = L_a \underline{e}_a}$$

Total Angular Momentum using Inertia Tensor

Rigid Body Motion Near Earth Surface: $(-m_i g \underline{k})$

Change in Angular Momentum

Consider how $\underline{J}$ changes

$$\dot{\underline{J}} = \sum_{i=1}^{N} m_i \underline{r}_i \times \ddot{\underline{r}}_i = \underbrace{\sum_{i=1}^{N} m_i \underline{r}_i}_{M\underline{R}} \times (-g\underline{k})$$

$$= -g\underline{R} \times \underline{k} M$$

$$\Rightarrow \boxed{\dot{\underline{J}} = -g\underline{R} \times \underline{k} M} \qquad (*am1)$$

Also

$$\dot{\underline{J}} = M\underline{R} \times \ddot{\underline{R}} + \dot{\underline{L}}, \qquad \boxed{\underline{L} = \sum_i m_i \underline{s}_i \times \dot{\underline{s}}_i}$$

$$= -Mg\underline{R} \times \underline{k} + \dot{\underline{L}} \qquad \left(\text{as } M\ddot{\underline{R}} = \underline{F}_{ext}^{(tot)} \Rightarrow M\ddot{\underline{R}} = -Mg\underline{k}\right)$$

(collection of particles, total internal forces is 0)

$$\Rightarrow -Mg\underline{R} \times \underline{k} = -Mg\underline{R} \times \underline{k} + \dot{\underline{L}} \qquad (\text{substituting } (*am1))$$

$$\Rightarrow \boxed{\dot{\underline{L}} = 0}, \quad (\text{equations relating components of } \underline{\omega})$$

Calculating $\dot{\underline{L}}$:

From $(*t3)$ we have that

$$\underline{L} = \underline{e}_a(t) I_{ab} \omega_b \Rightarrow \dot{\underline{L}} = \underline{e}_a(t) I_{ab} \dot{\omega}_b + \dot{\underline{e}}_a(t) I_{ab} \omega_b$$

time independant

$$\Rightarrow \dot{\underline{L}} = \underline{e}_a(t) I_{ab} \dot{\omega}_b + \varepsilon_{acd} \omega_d \underline{e}_c(t) I_{ab} \omega_b$$

$$\Rightarrow \dot{\underline{L}} = \underline{e}_c \left[I_{cb} \dot{\omega}_b + \varepsilon_{acd} \omega_d I_{ab} \omega_b \right]$$

dummy variable $a \leftrightarrow c$

Since from above, (if no external forces or in near earth gravity)

$$\dot{\underline{L}} = 0 \Rightarrow 0 = \underline{e}_c \left[I_{cb} \dot{\omega}_b + \varepsilon_{acd} \omega_d I_{ab} \omega_b \right]$$

In component form, this becomes

$$\boxed{I_{cb} \dot{\omega}_b + \varepsilon_{acd} \omega_d I_{ab} \omega_b = 0}$$

c is a free index and therefore, we have 3 sets of equations

Choose axes so that I_{ab} is a diagonal matrix. Therefore in terms of principal axes;

(principal moments of Inertia)

$$I_{ab} = \begin{cases} I_{11} & a=b=1 \\ I_{22} & a=b=2 \\ I_{33} & a=b=3 \\ 0 & a\neq b \end{cases}$$

c free index

i) c=1, $I_{11}\dot{\omega}_1 + \varepsilon_{a1d}\omega_d I_{ab}\omega_b = I_1\dot{\omega}_1 + \underbrace{\varepsilon_{213}}_{\substack{-1 \\ \text{odd permutation}}}\omega_3 I_{22}\omega_2 + \underbrace{\varepsilon_{312}}_{\substack{+1 \\ \text{even permutation}}}\omega_2 I_{33}\omega_3$

a=2, d=3   a=3 d=2

$$= I_{11}\dot{\omega}_1 - I_{22}\omega_3\omega_2 + I_{33}\omega_3\omega_2 = 0$$

$$\Rightarrow \boxed{I_{11}\dot{\omega}_1 - (I_{22}-I_{33})\omega_2\omega_3 = 0} \quad (*E1)$$

(ii) c=2

$$\Rightarrow \boxed{I_{22}\dot{\omega}_2 - (I_{33}-I_{11})\omega_1\omega_3 = 0} \quad (*E2)$$

(iii) c=3

$$\Rightarrow \boxed{I_{33}\dot{\omega}_3 - (I_{11}-I_{22})\omega_1\omega_2 = 0} \quad (*E3)$$

EULER'S EQUATIONS

Constants of Motion

(a) $\omega_1(*E1) + \omega_2(*E2) + \omega_3(*E_3) = I_{11}\dot{\omega}_1\omega_1 + I_{22}\dot{\omega}_2\omega_2 + I_{33}\dot{\omega}_3\omega_3 = 0$

$$\Rightarrow \frac{1}{2}\left(I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2\right) = \text{constant}$$

(b) $I_{11}\omega_1(*E1) + I_{22}\omega_2(*E2) + I_{33}\omega_3(*E3) = I_{11}^2\dot{\omega}_1\omega_1 + I_{22}^2\dot{\omega}_2\omega_2 + I_{33}\dot{\omega}_3\omega_3 = 0$

$$\Rightarrow I_{11}^2\omega_1^2 + I_{22}^2\omega_2^2 + I_{33}^2\dot{\omega}_3^2 = \text{constant}$$

Example: There is a solution where

$$\omega_1 = \omega_2 = 0 \quad , \quad \dot{\omega}_3 = 0 \quad , \quad \omega_3 \text{ is constant}$$

Is it stable?

Putting $\omega_1 = \eta_1 e^{pt}$, $\omega_2 = \eta e^{pt}$, $\omega_3 = \Omega + \eta_3 e^{pt}$

So
$$I_{33}\dot{\omega}_3 - (I_{11} - I_{22})\eta_1\eta_2 e^{2pt} + I_{33}\eta_3 p e^{pt} = 0$$

and further η_3 is second order small. ($\eta_1 \eta_2$ small)

$$\left(I_{11} p \eta_1 - (I_{22} - I_{33})\Omega \eta_3\right) e^{pt} = 0$$

$$\left(I_{22} p \eta_2 - (I_{33} - I_{11})\Omega \eta_1\right) e^{pt} = 0$$

i.e.
$$\begin{pmatrix} I_{11} p & -(I_{22} - I_{33})\Omega \\ -(I_{33} - I_{11})\Omega & I_{22} p \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \det \begin{pmatrix} I_{11} p & -(I_{22} - I_{33})\Omega \\ -(I_{33} - I_{11})\Omega & I_{22} p \end{pmatrix} = 0$$

(to want non-trivial solutions)

$$\Rightarrow \quad I_{11} I_{22} p^2 = \Omega^2 (I_{22} - I_{33})(I_{33} - I_{11})$$

Stable, then we want $p^2 < 0$ (so e^{pt} does not explode)

Stable $\Rightarrow p^2 < 0 \Rightarrow I_{22} > I_{33}$, $I_{33} < I_{11}$ or
$I_{22} < I_{33}$, $I_{33} > I_{11}$

Unstable $\Rightarrow p^2 > 0 \Rightarrow I_{22} > I_{33} > I_{11}$ or
$I_{11} > I_{33} > I_{22}$

growing solution

Example: When 2 of the moment of inertia are the same.

Suppose $I_{11} = I_{22}$ (eg: uniform cylinder)

Therefore

$$(*E3) \Rightarrow I_{33}\dot{\omega}_3 = 0 \Rightarrow \omega_3 \text{ is constant, let } \boxed{\omega_3 = \Omega}$$

$$(*E1) \Rightarrow I_{11}\dot{\omega}_1 + (I_{33} - I_{11})\omega_2\Omega = 0$$

$$(*E2) \Rightarrow I_{11}\dot{\omega}_2 + (I_{11} - I_{33})\omega_1\Omega = 0$$

So we can say that

$$(*d1) \quad \dot{\omega}_1 = -\Delta\Omega\omega_2 \qquad \text{where} \quad \boxed{\Delta = \frac{I_{33} - I_{11}}{I_{11}}}$$

$$(*d2) \quad \dot{\omega}_2 = \Delta\Omega\omega_1$$

Solving the pair of differential equations (*d1) and (*d2)

$$\dot{\omega}_1 = -\Delta\Omega\omega_2 \Rightarrow \ddot{\omega}_1 = -\Delta\Omega\dot{\omega}_2$$

$$\Rightarrow \ddot{\omega}_1 = -(\Delta\Omega)^2\omega_1$$

simple harmonic oscillator problem

↓

periodic solution

$$\Rightarrow \boxed{\omega_1 = A\cos|\Delta\Omega|t + B\sin|\Delta\Omega|t}$$

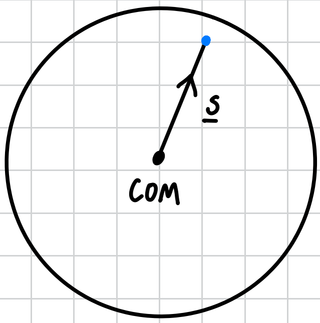
Similarly,

$$\boxed{\omega_2 = \frac{-1}{\Delta\Omega}\dot{\omega}_1}$$

Examples Computing Moments of Inertia

i) **Uniform mass density sphere:** mass density μ, Radius a

Total mass : $M = \frac{4}{3}\pi a^3 \mu$



In our notes above, moment of inertia tensor was done using sums, because we were concerned with discrete systems. Now, we have a continuum, so use integrals.

$$\boxed{I_{ab} = \int_{\text{sphere}} \left(|\underline{s}|^2\delta_{ab} - s_a s_b\right)\mu \, dV}$$

Take origin of co-ordinates from COM and use polar co-ordinates

$$s_1 = r\sin\theta\cos\phi$$
$$s_2 = r\sin\theta\sin\phi \qquad \text{and} \qquad r = |\underline{s}|$$
$$s_3 = r\cos\theta$$

$(\delta_{12}=0)$ $$I_{12} = -\mu \int_0^a r^2\,dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi\; r^2 \sin^2\theta \underbrace{\cos\phi \sin\phi}$$

↳ orthogonal functions ⇒ integral is 0

$$= 0$$

And similarly, we can show that

$$I_{12} = I_{23} = I_{31} = 0$$

Calculating a diagonal element

$$I_{33} = \mu \int_0^a r^2\,dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \left(r^2 \underbrace{\sin^2\theta}_{s_1^2+s_2^2}\right)$$

$$|s|^2\delta_{33} - s_3 s_3 = s_1^2 + s_2^2 + \cancel{s_3^2} - \cancel{s_3^2}$$
$$= s_1^2 + s_2^2$$

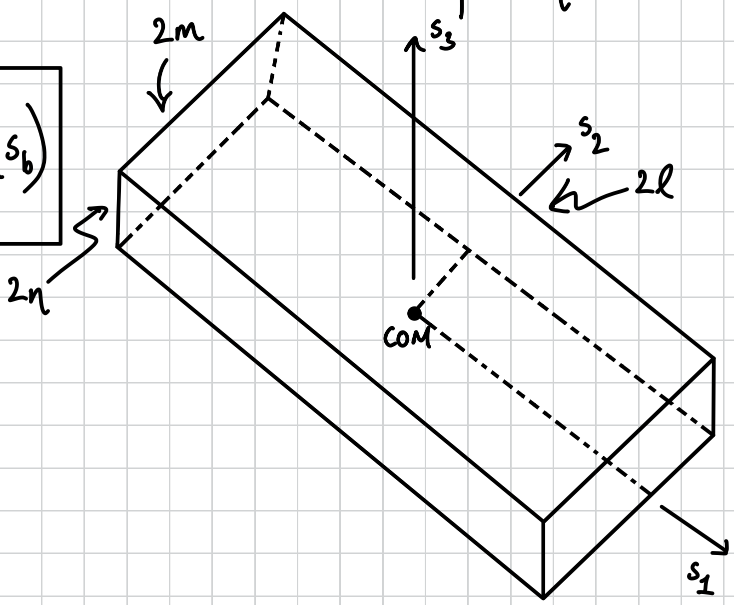
$$= 2\pi\mu \left[\frac{r^5}{5}\right]_0^a \underbrace{\int_0^{\pi} \sin^3\theta\, d\theta}_{=\frac{4}{3}}$$

$$= \frac{2\pi\mu a^5}{5}\,\frac{4}{3}$$

$$= M\frac{2}{5}a^2$$

$$\Rightarrow \boxed{I_{33} = M\frac{2}{5}a^2 = I_{22} = I_{11}}$$

ii) Constant density rectangular slab of material: mass density μ, width 2ℓ, breadth $2m$, depth $2n$



Rectangular slab of constant density

$$\boxed{I_{ab} = \mu \int_{-\ell}^{\ell} ds_1 \int_{-m}^{m} ds_2 \int_{-n}^{n} ds_3\; \delta_{ab}\left(s_1^2+s_2^2+s_3^2 - s_a s_b\right)}$$

For example calculating I_{12}

$$I_{12} = \mu \int_{-\ell}^{\ell} ds_1 \int_{-m}^{m} ds_2 \int_{-n}^{n} ds_3\, (-s_1 s_2)$$

$$= \mu\left[\frac{s_1^2}{2}\right]_{-\ell}^{\ell} + \cdots\cdots = 0 = I_{23} I_{13}$$

Calculating a diagonal element

$$I_{11} = \mu \int_{-l}^{l} ds_1 \int_{-m}^{m} ds_2 \int_{-n}^{n} ds_3 \left(s_2^2 + s_3^2 \right)$$

$$= \mu 2l \left(2n \left[\frac{s_2^3}{3} \right]_{-m}^{m} + 2m \left[\frac{s_3^3}{3} \right]_{-n}^{n} \right)$$

$$= \mu 2l \left(2n \, 2\frac{m^3}{3} + 2n \, 2\frac{n^3}{3} \right)$$

$$= \frac{\mu 8 lmn}{3} \left(m^2 + n^2 \right) = \frac{M}{3} \left(m^2 + n^2 \right)$$

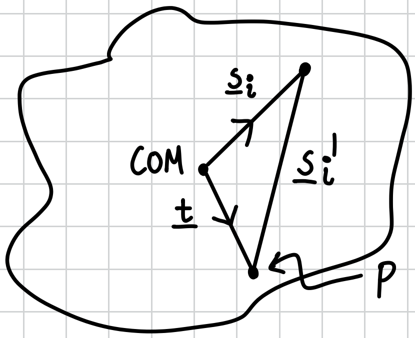
Similarly

$$I_{22} = \frac{M}{3} \left(l^2 + n^2 \right), \quad I_{33} = \frac{M}{3} \left(l^2 + m^2 \right)$$

iii) Moment of Inertia of another point shifted away from center of mass

Relative to P,

$$\underline{s}_i' = \underline{s}_i - t$$



$$I_{ab}' = \sum_i m_i \left(|\underline{s}_i'|^2 \delta_{ab} - s_{ia}' s_{ib}' \right)$$

$$= \sum_i m_i \left(|\underline{s}_i - \underline{t}|^2 \delta_{ab} - (\underline{s}_i - \underline{t})_a (\underline{s}_i - \underline{t})_b \right)$$

$$= \sum_i m_i \left(|\underline{s}_i| \delta_{ab} - s_{ia} s_{ib} \right) + M \left(|\underline{t}|^2 \delta_{ab} - t_a t_b \right)$$

(Note as shown before $\sum m_i s_i = 0$)

$$= I_{ab} + M \left(|\underline{t}|^2 \delta_{ab} - t_a t_b \right)$$

$$\Rightarrow \boxed{I_{ab}' = I_{ab} + M \left(|\underline{t}|^2 \delta_{ab} - t_a t_b \right)}$$

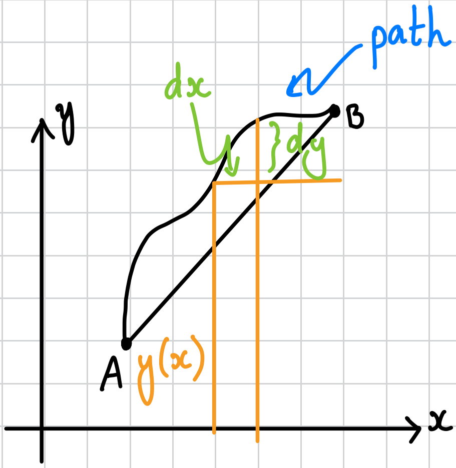
3. Lagrangian Dynamics

3.1 Calculus of Variations

Motivation: Trying to find the shortest distance/path between 2 points.

↳ such paths are called geodesics.

Consider a plane. The length of path A to B: L_{ab}



i) $$L_{ab} = \int_{x_A}^{x_B} \sqrt{1+y'^2}\, dx$$

ii) $$L_{ab} = \int_{s_A}^{s_B} \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}\, ds$$

We need to see the minimum of the integral: the shortest path

Typically on a surface (like a sphere) the element of length along a path will be

$$\int_{s_A}^{s_B} \sqrt{\alpha(x,y)\left(\frac{dx}{ds}\right)^2 + \beta(x,y)\left(\frac{dy}{ds}\right)^2}\, ds$$

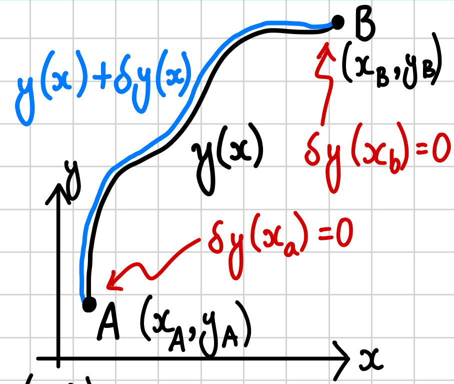
Considering stationary points of the quantities defined by integrals

$$\boxed{F[y] = \int_{x_A}^{x_B} f(x,y,y')\, dx}$$

, F[y] is called the functional

3.2 Euler-Lagrange Equations

Consider changing $$y(x) \longrightarrow y(x) + \delta y(x)$$



and then we look at

$$f(x, y+\delta y, y'+\delta y') = f(x,y,y') + \delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} + \cdots$$

multivariable Taylor's thm

$$= f(x,y,y') + \delta y \frac{\partial f}{\partial y} + \frac{d}{dx}\left[\delta y \frac{\partial f}{\partial y}\right] - \delta y \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \cdots$$

$$= f(x,y,y') + \delta y \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] + \frac{d}{dx}\left[\delta y \frac{\partial f}{\partial y'}\right] + \cdots$$

$$= f(x,y,y') + \delta y \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] + \frac{d}{dx}\left[\delta y \frac{\partial f}{\partial y'}\right] + \cdots\cdots$$

$$\Rightarrow F[y+\delta y] = \int_{x_A}^{x_B} \left(f(x,y,y') + \delta y \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] + \frac{d}{dx}\left[\delta y \frac{\partial f}{\partial y'}\right] + \cdots\cdots \right) dx$$

$$= F[y] + \int_{x_A}^{x_B} \partial y \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx + \cancel{\left[\delta y \frac{\partial f}{\partial y'}\right]_{x_A}^{x_B}} + \underbrace{\cdots\cdots}_{\text{higher order}}$$

$\delta y(x_A) = 0$ & $\delta y(x_B) = 0$

We can say that F[y] is stationary when

$$\boxed{\int_{x_A}^{x_B} \delta y \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx = 0} \Longrightarrow \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \quad (\text{* E-L})$$

function is stationary when first order $f'(x)$ is 0

(δy is arbitrary)

Euler-Lagrange Equation

- $f(x,y,y')$ is specified and Euler-Lagrange equation is really an equation for $y(x)$

Example: $f(x,y,y') = \sqrt{1-y'^2}$

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial f}{\partial x} \Rightarrow \text{Euler-Lagrange Eqn says}$$

$$\frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right) = 0 \Longrightarrow y' \text{ is constant}$$

Minimum Point of Euler-Lagrange Equations

Again using multivariable Taylor's theorem:

$$f(x, y+\delta y, y'+\delta y') = f(x,y,y') + \underbrace{\delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y}}_{\frac{d}{dx}\left(\delta y \frac{\partial f}{\partial y'}\right) - \delta y \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)} + \frac{1}{2}\delta y^2 \frac{\partial^2 f}{\partial y^2} + \underbrace{\delta y \delta y'}_{\frac{1}{2}(\delta y^2)'} \frac{\partial^2 f}{\partial y \partial y'} + \frac{1}{2}\delta y'^2 \frac{\partial^2 f}{\partial y'^2} + \dots$$

$$= f(x,y,y') + \delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y} + \frac{1}{2}\delta y^2 \frac{\partial^2 f}{\partial y^2} + \underbrace{\cancel{\frac{1}{2}\frac{d}{dx}\left(\delta y^2 \frac{\partial^2 f}{\partial y \partial y'}\right)}}_{\text{integrates to } 0 \text{ as } \delta y(x_A) = 0 = \delta y(x_B)} - \frac{1}{2}\delta y^2 \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial y'}\right) + \frac{1}{2}\delta y'^2 \frac{\partial^2 f}{\partial y'^2} + \dots$$

$$= f(x,y,y') + \delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y} + \frac{1}{2}\delta y^2 \underbrace{\left[\frac{\partial^2 f}{\partial y'^2} - \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial y'}\right)\right]}_{(*1)} + \frac{1}{2}(\delta y')^2 \underbrace{\frac{\partial^2 f}{\partial y'^2}}_{(*2)} + \dots$$

Provided $(*1)$ and $(*2)$ is positive, we get a minimum stationary point

> So for a minimum evaluated solution to E-L eqn;
>
> i) $\frac{\partial^2 f}{\partial y^2} - \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial y'}\right) > 0$
>
> ii) $\frac{\partial^2 f}{\partial y'^2} > 0$

Example: Check it for the shortest path between 2 points on a plane

3.3 Remarks

(a) If $f(x,y,y')$ is independent of y then

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \text{from } (* \text{E-L})$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial y'} \text{ is constant}}$$

(b) If $f(x,y,y')$ is independent of x i.e. $f(x,y,y')$ depend only on x via x and x')
then

no explicit dependance on x

$$\frac{\partial f}{\partial x} = 0$$

Then

$$\frac{d}{dx}f = \cancel{\frac{\partial f}{\partial x}\frac{dx}{dx}}^{=0} + \frac{\partial f}{\partial y}\frac{dy}{\partial x} + \frac{\partial f}{\partial y'}\frac{d^2y}{dx^2}$$

multivariable chain rule

product rule
$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$= y'\frac{\partial f}{\partial y} + \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) - y'\frac{d}{dx}\frac{\partial f}{\partial y'}$$

$$= y'\underbrace{\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)}_{(*\text{E-L}) = 0} + \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)$$

When y satisfies (*E-L) Euler Lagrange Equations, then

$$\frac{d}{dx}f = \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) \Rightarrow \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow \boxed{f - y'\frac{\partial f}{\partial y'} = \text{constant}}$$

(c) There could be several y's $(y_1, y_2, \dots y_n)$ and the functional would be

$$\boxed{F[y_1, \dots, y_n] = \int_a^b f(x, y_1, \dots, y_n, y_1', \dots, y_n')\, dx}$$

In this case, a stationary point corresponds to a set of (*E-L) Euler-Lagrange Equations

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0 \qquad \forall i \in \{1, \dots, N\}}$$

Note: If no explicit dependance on x, then

$$\frac{\partial f}{\partial x} = 0$$

and therefore

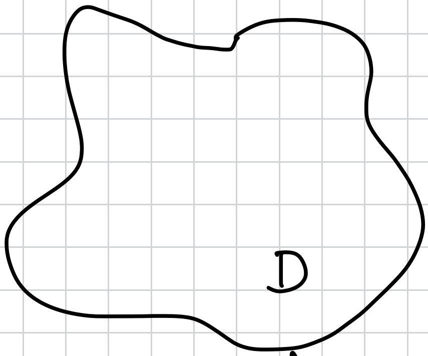
$$\frac{d}{dx}f = \cancel{\frac{\partial f}{\partial x}\frac{dx}{dx}}^{=0} + \sum_{i=1}^{N} y_i'\frac{\partial f}{\partial y_i} + \sum_{i=1}^{N} y_i''\frac{\partial f}{\partial y_i'}$$

$$\Rightarrow \frac{d}{dx}\left(f - \sum_{i=1}^{N} y_i'\frac{\partial f}{\partial y_i'}\right) = 0 \Rightarrow \boxed{f - \sum_{i=1}^{N} y_i'\frac{\partial f}{\partial y_i'} = \text{constant}}$$

d) The functional could be defined by a higher dimensional integral.

Define function $u(x,y)$, and the functional is

$$F[u] = \int_D f(x,y,u,\underbrace{u_x}_{\frac{\partial u}{\partial x}},\underbrace{u_y}_{\frac{\partial u}{\partial y}})\,dx\,dy$$



$\delta u = 0$ on boundary

The Euler-Lagrange equation in this case is

$$\boxed{\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial u_y} = 0} \qquad (* \text{ E-L2})$$

<u>Example</u>: $f = u_x^2 + u_y^2$. Then

$$F[u] = \int_D (u_x^2 + u_y^2)\,dx\,dy$$

Euler-Lagrange eqn is

$$-\frac{\partial}{\partial x}(2u_x) - \frac{\partial}{\partial y}(2u_y) = -2\underbrace{(u_{xx} - u_{yy})}_{\text{2D-Laplace eqn.}} = 0$$

3.4 Principle of Least Action

Suppose a dynamical system is described by a set of co-ordinates (generalized, could be a mix of Cartesian co-ordinates, polar co-ordinates e.t.c) which are called

$$\boxed{q_1(t), q_2(t), \dots, q_N(t)}$$

Then we also have a set of generalized velocities

$$\boxed{\dot{q}_1(t), \dot{q}_2(t), \dots, \dot{q}_N(t)}$$

So we have 2N variables

<u>Big idea</u>: The system moves or develops in time so that the action is stationary or minimal.

<u>Action</u>: The action A is a functional

$$A[q_1, q_2, \dots q_n] = \int_{t_1}^{t_2} \mathcal{L}(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)\,dt$$

$\mathcal{L}$ is called the Lagrangian.

The task is to choose the Lagrangian in such a way that the minimum of the action corresponds to the Newtonian equations of motion for the system expressed in terms of the generalized co-ordinates and their derivatives.

For the Lagrangian $\mathcal{L}$, we have a collection of (*E-L) Euler-Lagrange Equations,

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 \quad ; \quad i = 1, 2, \ldots, N} \qquad (*\text{E-L}3)$$

Definition: Generalized Momentum

Define generalized momentum p_i associated with the generalized co-ordinate q_i by setting

$$\boxed{p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}}$$

The generalised momentum conjugate to co-ordinate q_i can be substituted in (*E-L3),

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(p_i\right)}$$

Note: Neither the q_i or p_i need not be actual/specific components of a vector. They are generalised co-ordinates and momenta.

Remarks

a) If the Lagrangian does not explicitly on a generalized co-ordinate q_K, say, then

$$\frac{\partial \mathcal{L}}{\partial q_K} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_K}\right) = 0$$

$$\Rightarrow \frac{d}{dt}(p_K) = 0$$

$$\Rightarrow p_K = \text{CONSTANT}$$

Therefore if this happens, the co-ordinate q_K is ignorable and the associated generalized momentum p_K is conserved. In some circumstances, these can imply useful constraints of motion.

b) If the Lagrangian does not depend explicitly on t, then

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

and therefore

$$\frac{d\mathcal{L}}{dt} = \underbrace{\cancel{\frac{\partial \mathcal{L}}{\partial t}}}_{=0}\frac{dt}{dt} + \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial q_i} + \sum_{i=1}^{N} \ddot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

multivariable chain rule

$$= \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial q_i} + \sum_{i=1}^{N}\left[\frac{d}{dt}\left(\dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \dot{q}_i \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right]$$

product rule

$$= \sum_{i=1}^{N} \dot{q}_i \left[\underbrace{\cancel{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)}}_{=0}\right] + \sum_{i=1}^{N} \frac{d}{dt}\left(\dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

Euler-Lagrange eqn
(*E-L3)

$$= \frac{d}{dt} \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Therefore we get that

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt} \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Rightarrow \frac{d}{dt}\left(\mathcal{L} - \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$

$$\Rightarrow \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = \text{CONSTANT}$$

Note: The conserved quantity

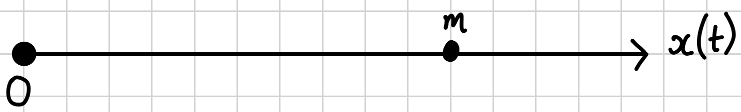
$$\boxed{J(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) \equiv \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}}$$

is called the Jacobi function and is often equal to the total conserved energy.

3.5 Examples

Example 1: 1D System

Particle of mass m moving along the x-axis



subject to a force given by a potential $V(x)$

$$m\ddot{x} = -\frac{dV(x)}{dx}$$

A suitable Lagrangian is

$$\mathcal{L}(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x)$$

To check, we simply calculate the Euler-Lagrange equation for x:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad , \quad \frac{\partial \mathcal{L}}{\partial x} = -\frac{dV}{dx}$$

to find from (*E-L3)

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = -\frac{dV}{dx} - \frac{d}{dt}(m\dot{x})$$

$$= -\frac{dV}{dx} - m\ddot{x} = 0$$

$$\Longrightarrow \quad m\ddot{x} = -\frac{dV}{dx}$$ Equation of motion.

Here V does not depend on t: $\frac{dV}{dt} = 0$

Note: The Lagrangian depends on $x(t)$ and $\dot{x}(t)$ but no explicit dependance on time.

This means by remark 3.4 b) ;

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \mathcal{L} = \text{CONSTANT}$$

$$\Rightarrow \dot{x}(m\dot{x}) - \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \text{CONSTANT}$$

$$\Rightarrow m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + V(x) = \text{CONSTANT}$$

$$\Rightarrow \frac{1}{2}m\dot{x}^2 + V(x) = \text{CONSTANT} = E$$

Kinetic energy potential energy Total Energy

Note: It is important to observe that

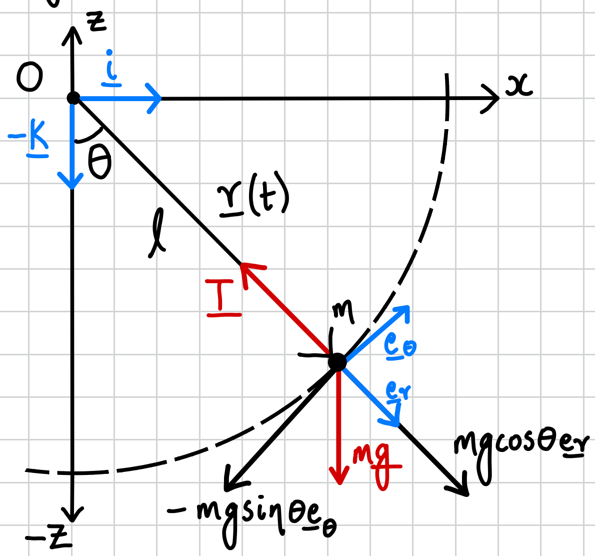
$$\mathcal{L}(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x) = K.E - P.E$$

In general, we can say that

$$\boxed{\mathcal{L} = KE - PE} \qquad (*\mathcal{L}1)$$

Example 2: A Simple Pendulum

Consider the following diagram (pendulum of mass m)



$\theta = \theta(t)$

The pendulum is a particle of mass m is constrained to move in a plane with unit vectors $\underline{k}$ and $\underline{i}$ at the end of a light rod of length l

Assume inextensible

i.e. assume it has 0 mass

Newton's equation of motion is therefore

$$m\ddot{\underline{r}} = -mg\underline{k} + \underline{T}$$

($\underline{T}$ is part of the solution)

The position $\underline{r}(t)$, velocity $\dot{\underline{r}}(t)$ and acceleration $\ddot{\underline{r}}(t)$ is given by

$$\underline{r}(t) = l\sin\theta\,\underline{i} - l\cos\theta\,\underline{k} \implies \dot{\underline{r}}(t) = l\dot{\theta}(\cos\theta\,\underline{i} + \sin\theta\,\underline{k})$$

$$\implies \ddot{\underline{r}}(t) = l\ddot{\theta}(\cos\theta\,\underline{i} + \sin\theta)\underline{k} - l\dot{\theta}^2(\sin\theta\,\underline{i} - \cos\theta\,\underline{k})$$

A Lagrangian for this system involves only K.E and P.E.

- The Kinetic Energy K.E. is

$$K.E = \frac{1}{2}m|\dot{\underline{r}}|^2 \implies \boxed{K.E = \frac{1}{2}ml^2\dot{\theta}^2}$$

- The Potential Energy P.E is

$$V(\theta) = mgz \implies \boxed{V(\theta) = -mgl\cos\theta}$$

Therefore, we can write the Lagrangian as

$$\mathcal{L}(\theta, \dot{\theta}) = K.E - P.E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

Using this Lagrangian, Euler-Lagrange Equation (*E-L3) for θ is

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \Rightarrow -mgl\sin\theta - \frac{d}{dt}\left(ml^2\dot{\theta}\right) =$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta$$

For simple harmonic motion, θ is small $\Rightarrow \sin\theta \approx \theta$;

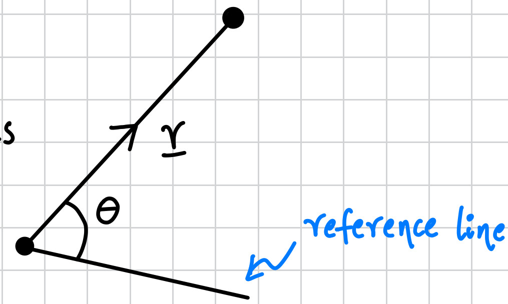
$$\ddot{\theta} \approx -\frac{g}{l}\theta$$

This formulation allows a calculation of θ directly.

Moreover, the Lagrangian has no specific dependance on t, so by similar arguments,

$$K+V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = \text{CONSTANT} = E$$

Example 3: Particle in a Plane



Consider a particle of mass m moving on a plane that is subject to a force with potential $V(\underline{r})$ directed towards the origin.

Using plane polar co-ordinates:

$$\underline{r} = -r\underline{e_r} \Rightarrow \dot{\underline{r}} = -\left(\dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta}\right)$$

$$\Rightarrow |\dot{\underline{r}}|^2 = \left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Therefore the Kinetic Energy is

$$K.E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Then the Lagrangian will be

$$\mathcal{L}(r, \dot{r}, \theta, \dot{\theta}) = K - V$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

Notice that the Lagrangian does not depend on the co-ordinate θ, and so

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant}$$ (remark 3.4 a)

Also the Euler-Lagrange for θ is (Lagrangian independant of θ)

$$\underbrace{\cancel{\frac{\partial \mathcal{L}}{\partial \theta}}}_{=0} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \text{CONSTANT}$$

$$\Rightarrow p_\theta = mr^2\dot{\theta} = \text{CONSTANT}$$

$$\Rightarrow p_\theta = mh = \text{CONSTANT}, \quad \text{put } \boxed{r^2\dot{\theta} = h}$$

Also the Euler-Lagrange equation for r is

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = mr\dot{\theta}^2 - \frac{dV}{dr} - m\ddot{r} = 0 \Rightarrow m\ddot{r} - mr\dot{\theta}^2 = -\frac{dV}{dr}$$

Hence putting $r^2\dot{\theta} = h$ as previously, the equation for r is

$$m\ddot{r} = \frac{mh^2}{r^3} - \frac{dV}{dr}$$

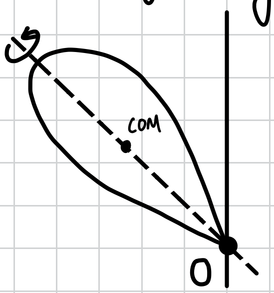
Furthermore the Lagrangian does not depend explicitly on t, which means

$$\dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} + \dot{\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} = m\dot{r}^2 + mr^2\dot{\theta}^2 - \left(\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\right)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \quad = E = \text{CONSTANT}$$

3.6 Rigid Bodies - Spinning Top

Consider the following diagram

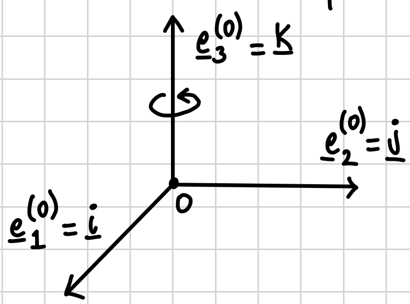


When we have a spinning top, we have 3 angles to describe the top.

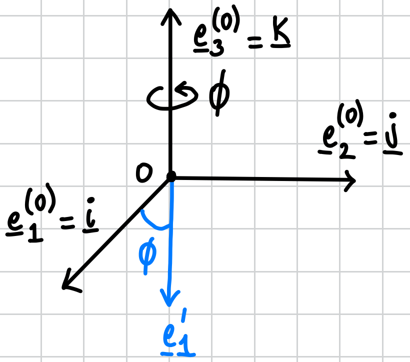
First, we need to find a way to specify position of the top relative to the fixed point O.

a) Euler Angles

Suppose the fixed axes are set up so that the vertical is in the direction of the unit vector $e_3^{(0)}$ (unit vector $\underline{k}$) and other 2 are plane perpendicular to the vertical one.

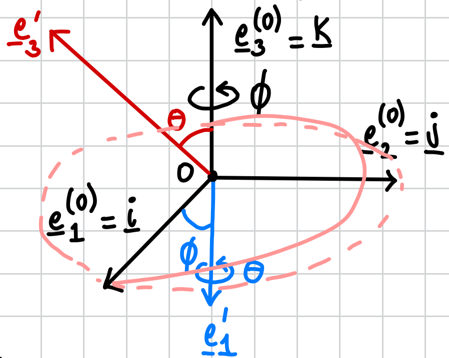


i) Rotate around vertical axis an angle of ϕ:
Let R_ϕ represent an anticlockwise rotation around the axis $\underline{e}_3^{(0)}$ through an angle ϕ



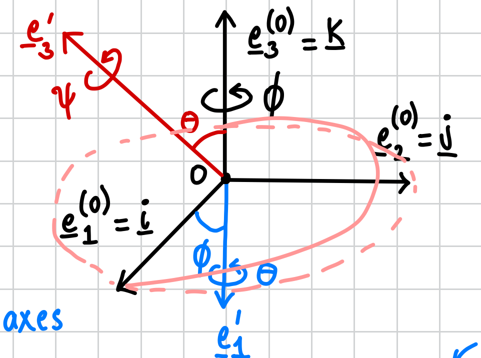
ii) Under rotation i), the axis in the direction $\underline{e}_1^{(0)}$ is rotated to a new position say $\underline{e}_1'$

iii) Next is to rotate through an angle θ around the new $\underline{e}_1'$. Let R_θ represent an anti-clockwise rotation around the axis $\underline{e}_1'$ through an angle θ



iv) Under the rotation iii), $\underline{e}_3^{(0)}$ is rotated to a new position, say $\underline{e}_3^{(0)}$ and it is useful to choose this as the symmetry of the top.

v) Next step is to let R_ψ represent an anticlockwise rotation around the axis $\underline{e}_3'$ through an angle ψ



fixed axes

principal axes

So the original axis $\underline{e}_1^{(0)}, \underline{e}_2^{(0)}, \underline{e}_3^{(0)}$ are rotated to $\underline{e}_1, \underline{e}_2, \underline{e}_3$ fixed in the top (symmetrical) and $\underline{e}_3$ is the axis of symmetry.

Thus combining these changes from the fixed axes to principal axes of the spinning top is represented by

$$\boxed{R = R_\psi R_\theta R_\phi}$$

The angles ϕ, θ and ψ are Euler Angles.

Looking back at Rigid Body section 2.13, we identify the 3 components of angular velocity ω by calculating

$$\dot{R}R^T$$

Differentiating R using product rule;

$$\dot{R} = (\dot{R}_\phi R_\theta R_\psi + R_\phi \dot{R}_\theta R_\psi + R_\phi R_\theta \dot{R}_\psi)$$

and calculating transpose

$$R^T = (R_\phi R_\theta R_\psi)^T = R_\psi^T R_\theta^T R_\phi^T$$

$$\left[\text{Check } RR^T = R_\phi R_\theta \underbrace{R_\psi R_\psi^T}_{\mathbb{1}} R_\theta^T R_\phi^T = \mathbb{1}\right]$$

Therefore we get that

$$\dot{R}R^T = (\dot{R}_\phi R_\theta R_\psi + R_\phi \dot{R}_\theta R_\psi + R_\phi R_\theta \dot{R}_\psi)(R_\psi^T R_\theta^T R_\phi^T)$$

using that $RR^T = \mathbb{1}$ ← identity matrix

$$\Rightarrow \boxed{\dot{R}R^T = \dot{R}_\psi R_\psi^T + R_\psi \dot{R}_\theta R_\theta^T R_\psi^T + R_\psi R_\theta \dot{R}_\phi R_\phi^T R_\theta^T R_\psi^T} \quad (*R)$$

To finish the computation requires explicit expression for the rotation matrices. These are

$$\boxed{R_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \qquad R_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Therefore we get that

$$\dot{R}_\psi R_\psi^T = \begin{pmatrix} -\sin\psi & \cos\psi & 0 \\ -\cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\psi} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \dot{\psi} & 0 \\ -\dot{\psi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \boxed{\dot{R}_\psi R_\psi^T = \begin{pmatrix} 0 & \dot{\psi} & 0 \\ -\dot{\psi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

Similarly

$$\boxed{\dot{R}_\theta R_\theta^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot{\theta} \\ 0 & -\dot{\theta} & 0 \end{pmatrix}}, \quad \boxed{\dot{R}_\psi R_\psi = \begin{pmatrix} 0 & \dot{\phi} & 0 \\ -\dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

And by substitution into (*R) $\dot{R}R^T$ we get

$$\boxed{\dot{R}R^T = \begin{pmatrix} 0 & \dot{\psi}+\dot{\phi}\cos\theta & \dot{\theta}\sin\psi - \dot{\phi}\cos\psi\sin\theta \\ -\dot{\psi}-\dot{\phi}\cos\theta & 0 & \dot{\theta}\cos\psi + \dot{\phi}\sin\psi\sin\theta \\ -\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\sin\theta & -\dot{\theta}\cos\psi - \dot{\phi}\sin\psi\sin\theta & 0 \end{pmatrix}} \quad (*R2)$$

and from section 2.13 Rigid Bodies, $\dot{R}R^T$ is antisymmetric, and hence

$$(\dot{R}R^T)_{ab} = \varepsilon_{abc}\, w_c(t)$$

angular velocities

and therefore

$$\boxed{\dot{R}R^T = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}} \quad (*R3)$$

Comparing (*R2) and (*R3) to get

$$w_1 = \dot{\theta}\cos\psi + \dot{\theta}\sin\psi\sin\theta, \quad w_2 = -\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\sin\theta, \quad w_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

Now deriving Kinetic Energy:

axes of symmetry

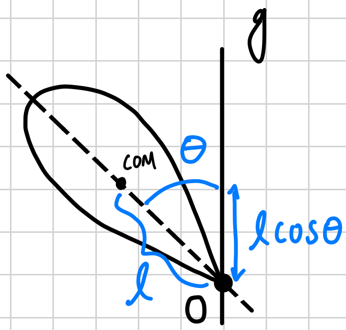
Using the principal moments of Inertia $I_{11} = I_{22} \equiv A$, $I_{33} \equiv C$

$$K.E = \frac{1}{2}\left(I_{33}w_3^2 + I_{11}(w_1^2 + w_2^2)\right)$$

$$\Rightarrow \boxed{K.E = \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{1}{2}A(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)}$$

Now deriving potential energy:

For a typical symmetrical top, the center of mass is a distance l from the fixed point and situated along the symmetry axis.



If top has mass M,

$$\boxed{V = Mgl\cos\theta}$$

Therefore putting it all together, we get the Lagrangian for the symmetric top as

$$\mathcal{L}(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}) = K - V$$

$$\Rightarrow \boxed{\mathcal{L}(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}) = \frac{1}{2}\left(C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + A\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)\right) - Mgl\cos\theta}$$